

Optical Waveguide Theory (C)



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... ?

“This concerns time harmonic fields ... with angular frequency ... ,
for vacuum wavenumber ... , speed of light ... , and wavelength”

“The problem is governed by the Maxwell curl equations in the
frequency domain for the electric field ... and magnetic field ... , for
(lossless) uncharged dielectric, nonmagnetic linear (isotropic) media
with (piecewise constant) relative permittivity ... :

... (.)”

[M. Hammer, A. Hildebrandt, J. Förstner, *Journal of Lightwave Technology* **34**(3), 997 (2016)]

Course overview

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Hybrid analytical / numerical coupled mode theory.
 - Oblique semi-guided waves: 2-D integrated optics.

Maxwell equations, Fourier transform

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}}$$

$$\& \quad \mathbf{F}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int \tilde{\mathbf{F}}(\mathbf{r}, \omega) e^{i\omega t} d\omega, \quad \tilde{\mathbf{F}}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int \mathbf{F}(\mathbf{r}, t) e^{-i\omega t} dt$$

$$\begin{aligned} &\curvearrowright \quad \mathbf{E}(\mathbf{r}, t), \mathbf{D}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t), \mathbf{H}(\mathbf{r}, t), \rho_f(\mathbf{r}, t), \mathbf{J}_f(\mathbf{r}, t) \\ &\longleftrightarrow \quad \tilde{\mathbf{E}}(\mathbf{r}, \omega), \tilde{\mathbf{D}}(\mathbf{r}, \omega), \tilde{\mathbf{B}}(\mathbf{r}, \omega), \tilde{\mathbf{H}}(\mathbf{r}, \omega), \tilde{\rho}_f(\mathbf{r}, \omega), \tilde{\mathbf{J}}_f(\mathbf{r}, \omega), \end{aligned}$$

$$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_f, \quad \nabla \times \tilde{\mathbf{E}} = -i\omega \tilde{\mathbf{B}}, \quad \nabla \cdot \tilde{\mathbf{B}} = 0, \quad \nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}}_f + i\omega \tilde{\mathbf{D}}$$

(Caution: arbitrary choice of $\sim \exp(\pm i\omega t)$!).

Maxwell equations, frequency domain

$$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_f, \quad \nabla \times \tilde{\mathbf{E}} = -i\omega\tilde{\mathbf{B}}, \quad \nabla \cdot \tilde{\mathbf{B}} = 0, \quad \nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}}_f + i\omega\tilde{\mathbf{D}}.$$

$$\mathbf{F}(\mathbf{r}, t) \in \mathbb{R} \rightsquigarrow \tilde{\mathbf{F}}(\mathbf{r}, -\omega) = (\tilde{\mathbf{F}}(\mathbf{r}, \omega))^*$$

“at frequency ω_0 ”: $\tilde{\mathbf{F}}(\mathbf{r}, \omega) = \sqrt{\frac{\pi}{2}} \bar{\mathbf{F}}(\mathbf{r}) \delta(\omega - \omega_0) + \sqrt{\frac{\pi}{2}} \bar{\mathbf{F}}^*(\mathbf{r}) \delta(\omega + \omega_0)$

$$\curvearrowright \mathbf{F}(\mathbf{r}, t) = \frac{1}{2} \left\{ \bar{\mathbf{F}}(\mathbf{r}) e^{i\omega_0 t} + \bar{\mathbf{F}}^*(\mathbf{r}) e^{-i\omega_0 t} \right\},$$

$$\mathbf{F}(\mathbf{r}, t) = \text{Re} \left\{ \bar{\mathbf{F}}(\mathbf{r}) e^{i\omega_0 t} \right\},$$

$$\text{“}\mathbf{F}(\mathbf{r}, t) = \frac{1}{2} \bar{\mathbf{F}}(\mathbf{r}) e^{i\omega_0 t} + \text{c.c.}\text{”}.$$

$$\curvearrowright \bar{\mathbf{E}}(\mathbf{r}), \bar{\mathbf{D}}(\mathbf{r}), \bar{\mathbf{B}}(\mathbf{r}), \bar{\mathbf{H}}(\mathbf{r}), \bar{\rho}_f(\mathbf{r}), \bar{\mathbf{J}}_f(\mathbf{r}), \sim \exp(i\omega_0 t),$$

$$\nabla \cdot \bar{\mathbf{D}} = \bar{\rho}_f, \quad \nabla \times \bar{\mathbf{E}} = -i\omega_0 \bar{\mathbf{B}}, \quad \nabla \cdot \bar{\mathbf{B}} = 0, \quad \nabla \times \bar{\mathbf{H}} = \bar{\mathbf{J}}_f + i\omega_0 \bar{\mathbf{D}}.$$

Caution: Decorations $\sim, \bar{\cdot}, \hat{\cdot}$ are usually omitted; context determines interpretation of symbols.

Magnetization

$\tilde{\mathbf{M}}$: density of magnetic dipole moments (bound currents).

$$\tilde{\mathbf{H}} = \frac{1}{\mu_0} \tilde{\mathbf{B}} - \tilde{\mathbf{M}}, \quad [\tilde{\mathbf{H}}] = [\mathbf{M}] = \frac{\text{A m}^2}{\text{m}^3}, \quad [\tilde{\mathbf{B}}] = \text{T} = \frac{\text{Vs}}{\text{m}^2},$$

▶ vacuum permeability $\mu_0 = 4\pi \cdot 10^{-7} \left[\frac{\text{N}}{\text{A}^2} = \frac{\text{Vs}}{\text{Am}} \right]$.

• Local dipoles induced by $\tilde{\mathbf{H}} \rightsquigarrow \tilde{\mathbf{M}}(\tilde{\mathbf{H}})$.

• Linear magnetic media:

$$\tilde{\mathbf{M}} = \hat{\chi}_m \tilde{\mathbf{H}}, \quad \hat{\chi}_m: \text{magnetic susceptibility}, \quad [\hat{\chi}_m] = \hat{1}.$$

$$\curvearrowright \tilde{\mathbf{B}} = \mu_0 (\hat{1} + \hat{\chi}_m) \tilde{\mathbf{H}} = \mu_0 \hat{\mu} \tilde{\mathbf{H}}, \quad \hat{\mu}: \text{relative permeability}, \quad [\hat{\mu}] = \hat{1}.$$

• $\hat{\chi}_m(\mathbf{r}, \omega), \hat{\mu}(\mathbf{r}, \omega)$ are determined in the frequency domain.

• Complications: manifold.

• Traditional integrated optics (frequencies, media): $\hat{\mu}(\boldsymbol{\neq}) = \hat{1}$.

Polarization

$\tilde{\mathbf{P}}$: density of electric dipole moment (bound charges).

$$\tilde{\mathbf{D}} = \epsilon_0 \tilde{\mathbf{E}} + \tilde{\mathbf{P}}, \quad [\tilde{\mathbf{D}}] = [\tilde{\mathbf{P}}] = \frac{\text{As m}}{\text{m}^3}, \quad [\tilde{\mathbf{E}}] = \frac{\text{V}}{\text{m}},$$

▶ vacuum permittivity $\epsilon_0 = 8.854187817 \dots \cdot 10^{-12} \left[\frac{\text{F}}{\text{m}} = \frac{\text{As}}{\text{Vm}} \right]$.

• Local dipoles induced by $\tilde{\mathbf{E}} \rightsquigarrow \tilde{\mathbf{P}}(\tilde{\mathbf{E}})$.

• Linear dielectrics:

$$\tilde{\mathbf{P}} = \epsilon_0 \hat{\chi}_e \tilde{\mathbf{E}}, \quad \hat{\chi}_e: \text{dielectric susceptibility}, \quad [\hat{\chi}_e] = \hat{1}.$$

$$\curvearrowright \tilde{\mathbf{D}} = \epsilon_0 (\hat{1} + \hat{\chi}_e) \tilde{\mathbf{E}} = \epsilon_0 \hat{\epsilon} \tilde{\mathbf{E}}, \quad \hat{\epsilon}: \text{relative permittivity}, \quad [\hat{\epsilon}] = \hat{1}.$$

• $\hat{\chi}_e(\mathbf{r}, \omega), \hat{\epsilon}(\mathbf{r}, \omega)$ are determined in the frequency domain.

• Complications: $\text{Im } \epsilon, \hat{\epsilon}(T), \hat{\epsilon}(\mathbf{F}), \chi_{jkl}^{(2)} E_k E_l, \chi_{jklm}^{(3)} E_k E_l E_m, \dots$

• Simpler cases: $\hat{\epsilon}(\boldsymbol{\neq}), \hat{\epsilon} = \epsilon \hat{1}$.

Maxwell equations, dispersion

(Material) dispersion: $\hat{\epsilon}(\mathbf{r}, \omega), \hat{\mu}(\mathbf{r}, \omega)$ are frequency dependent.

$$\tilde{\mathbf{D}}(\mathbf{r}, \omega) = \epsilon_0 \hat{\epsilon}(\mathbf{r}, \omega) \tilde{\mathbf{E}}(\mathbf{r}, \omega), \quad \tilde{\mathbf{B}}(\mathbf{r}, \omega) = \mu_0 \hat{\mu}(\mathbf{r}, \omega) \tilde{\mathbf{H}}(\mathbf{r}, \omega)$$

$$\curvearrowright \mathbf{D}(\mathbf{r}, t) = \epsilon_0 \int \hat{\epsilon}_{\text{TD}}(\mathbf{r}, t - t') \mathbf{E}(\mathbf{r}, t') dt',$$

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 \int \hat{\mu}_{\text{TD}}(\mathbf{r}, t - t') \mathbf{H}(\mathbf{r}, t') dt'.$$

Helmholtz equations

Linear dielectric media without free charges or currents,
time dependence $\sim \exp(i\omega t)$, fields $\mathbf{E}(\mathbf{r})$, $\mathbf{D}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$,
material properties $\hat{\epsilon}(\mathbf{r})$, $\hat{\mu}(\mathbf{r})$:

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -i\omega \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = i\omega \mathbf{D},$$

$$\mathbf{D} = \epsilon_0 \hat{\epsilon} \mathbf{E}, \quad \mathbf{B} = \mu_0 \hat{\mu} \mathbf{H}.$$

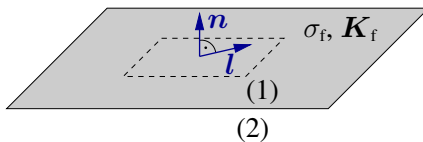
$$\nabla \times \mathbf{E} = -i\omega \mu_0 \hat{\mu} \mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega \epsilon_0 \hat{\epsilon} \mathbf{E}, \quad \nabla \cdot \hat{\epsilon} \mathbf{E} = 0, \quad \nabla \cdot \hat{\mu} \mathbf{H} = 0.$$

$$\nabla \times (\hat{\mu}^{-1} \nabla \times \mathbf{E}) = \omega^2 \epsilon_0 \mu_0 \hat{\epsilon} \mathbf{E} \quad \text{or} \quad \nabla \times (\hat{\epsilon}^{-1} \nabla \times \mathbf{H}) = \omega^2 \epsilon_0 \mu_0 \hat{\mu} \mathbf{H}.$$

Where $\hat{\epsilon} = \epsilon \hat{1}$, $\nabla \epsilon = 0$, $\hat{\mu} = \mu \hat{1}$, $\nabla \mu = 0$: (!)

$$\Delta \mathbf{E} + \frac{\omega^2}{c^2} \epsilon \mu \mathbf{E} = 0 \quad \text{or} \quad \Delta \mathbf{H} + \frac{\omega^2}{c^2} \epsilon \mu \mathbf{H} = 0, \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

Interface conditions



Surface between media (1) and (2), surface normal \mathbf{n} , tangents \mathbf{l} ,
surface charge density σ_f , surface current density \mathbf{K}_f :

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \sigma_f, \quad \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

$$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{l} \cdot (\mathbf{K}_f \times \mathbf{n}).$$

Plane harmonic waves

Where $\hat{\epsilon} = \epsilon \hat{1}$, $\nabla \epsilon = 0$, $\hat{\mu} = \mu \hat{1}$, $\nabla \mu = 0$: (!)
Components of \mathbf{E} , \mathbf{H} satisfy

$$\psi(\mathbf{r}, t) = \psi_0 e^{-i(\mathbf{k}_m \cdot \mathbf{r} - \omega t)}, \quad \Delta \psi + \frac{\omega^2}{c^2} \epsilon \mu \psi = 0,$$

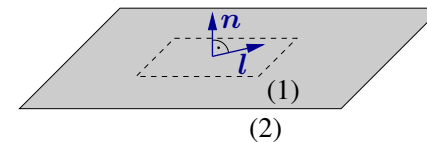
$$-k_m^2 + \frac{\omega^2}{c^2} \epsilon \mu = 0.$$

(Mixture of TD and FD expressions: $\nabla \cdot \nabla = \nabla^2$, Re., 1/2, c.c. omitted; sloppy, but common.)

- | | | |
|------------------------|---------------------------|--|
| • Medium: | refractive index: | $n = \sqrt{\epsilon \mu}$ |
| • Periodicity in time: | angular frequency: | $\omega,$ |
| | frequency: | $f = \omega / (2\pi),$ |
| | period: | $T = 1/f = 2\pi / \omega,$ |
| • Spatial periodicity: | wave vector: | $\mathbf{k}_m, k_m = \mathbf{k}_m ,$ |
| | wavenumber: | $k_m = \omega / c_m = (\omega / c) n = k n,$ |
| | vacuum wavenumber: | $k = \omega / c,$ |
| | vacuum wavelength: | $\lambda = 2\pi / k = 2\pi c / \omega,$ |
| | wavelength in the medium: | $\lambda_m = 2\pi / k_m = 2\pi / (k n) = \lambda / n.$ |
| • Phase velocity: | speed of light in vacuum: | $c = 1 / \sqrt{\epsilon_0 \mu_0} = \lambda f,$ |
| | in the medium: | $c_m = c / n = \lambda_m f.$ |
- (Use of symbols depends highly on context.)

Electromagnetic spectrum ▶ ▶

Interface conditions

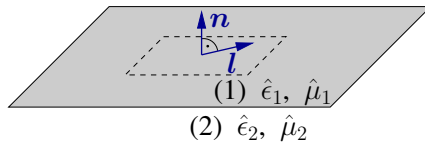


Surface between media (1) and (2), surface normal \mathbf{n} , tangents \mathbf{l} ,
surface without free charges or currents:

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

$$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = 0.$$

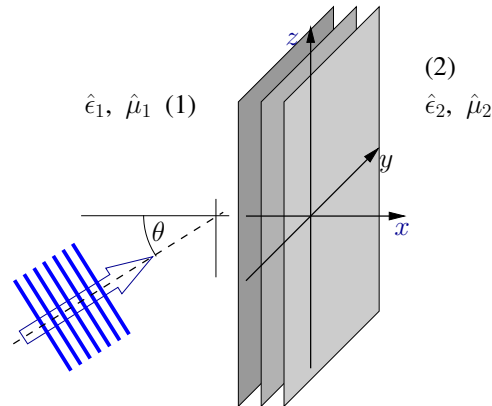
Interface conditions



Surface between media (1) and (2), surface normal \mathbf{n} , tangents \mathbf{l} , linear media with permittivities $\hat{\epsilon}_1, \hat{\epsilon}_2$, and permeabilities $\hat{\mu}_1, \hat{\mu}_2$:

$$\begin{aligned} \mathbf{n} \cdot (\hat{\epsilon}_1 \mathbf{E}_1 - \hat{\epsilon}_2 \mathbf{E}_2) &= 0, & \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) &= 0, \\ \mathbf{n} \cdot (\hat{\mu}_1 \mathbf{H}_1 - \hat{\mu}_2 \mathbf{H}_2) &= 0, & \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) &= 0. \end{aligned}$$

Dielectric multilayer structures



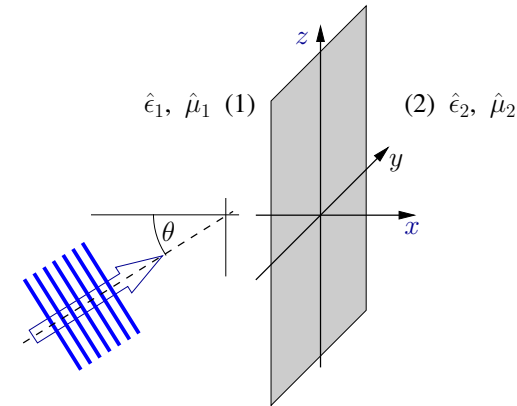
$$\begin{aligned} \nabla \times \mathbf{E} &= -i\omega\mu_0\hat{\mu}\mathbf{H}, \\ \nabla \times \mathbf{H} &= i\omega\epsilon_0\hat{\epsilon}\mathbf{E} \\ \hat{\epsilon}(\mathbf{r}) \text{ and } \hat{\mu}(\mathbf{r}) &\text{ are constant along } y, z \end{aligned}$$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}'(x) e^{-i(k_y y + k_z z)}, \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}'(x) e^{-i(k_y y + k_z z)}$$

1-D problem for \mathbf{E}' , \mathbf{H}' .

Reflectance and transmittance properties.

Reflection and transmission of plane waves at dielectric interfaces



(FD)

$$\begin{aligned} \nabla \times \mathbf{E} &= -i\omega\mu_0\hat{\mu}\mathbf{H}, \\ \nabla \times \mathbf{H} &= i\omega\epsilon_0\hat{\epsilon}\mathbf{E} \\ \hat{\epsilon}(\mathbf{r}) \text{ and } \hat{\mu}(\mathbf{r}) &\text{ are constant along } y, z \end{aligned}$$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}'(x) e^{-i(k_y y + k_z z)}, \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}'(x) e^{-i(k_y y + k_z z)}$$

1-D problem for \mathbf{E}' , \mathbf{H}' .

(incoming plane wave at angle θ)
(orient coordinates ($k_y = 0$), plane of incidence, distinguish polarizations)
(write ansatz functions for incoming, reflected, and transmitted waves)
(interface conditions determine the amplitudes)

Fresnel equations.

Energy of electromagnetic fields

(TD)

- Force on a particle with charge q , velocity \mathbf{v} , in a field \mathbf{E}, \mathbf{B} :
 $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$,
- work for shifting the particle by $d\mathbf{r} = \mathbf{v} dt$:
 $dW = \mathbf{F} \cdot d\mathbf{r} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} dt = q\mathbf{E} \cdot \mathbf{v} dt$,
- respective power: $\frac{dW}{dt} = q\mathbf{E} \cdot \mathbf{v}$.

For a charge density $\rho_f(\mathbf{r}, t)$:

- force density $\mathbf{f} = \rho_f(\mathbf{E} + \mathbf{v} \times \mathbf{B})$,
- power density $\mathbf{f} \cdot \mathbf{v} = \rho_f \mathbf{E} \cdot \mathbf{v} = \mathbf{J}_f \cdot \mathbf{E}$,

$$\text{total work per time unit done in } \mathcal{V}: \quad \frac{dW_{\mathcal{V}}}{dt} = \int_{\mathcal{V}} \mathbf{J}_f \cdot \mathbf{E} dV.$$

Power & energy density, Poynting theorem

(TD)

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}}, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}$$

$$\hookrightarrow \frac{d}{dt} W_V^{\text{mech}} = \int_V \mathbf{J}_f \cdot \mathbf{E} dV = - \int_V (\mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}}) dV - \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dV,$$

- Poynting vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, (energy flux density, power density)
- energy density: $w = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$, $W_V^{\text{field}} = \int_V w dV$,
- $\hat{\epsilon}^\dagger = \hat{\epsilon}$, $\hat{\epsilon}(\omega)$, $\mathbf{D} = \epsilon_0 \hat{\epsilon} \mathbf{E}$, $\hat{\mu}^\dagger = \hat{\mu}$, $\hat{\mu}(\omega)$, $\mathbf{B} = \mu_0 \hat{\mu} \mathbf{H}$ (!)
 $\rightsquigarrow \dot{w} = (\mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}})$

$$\hookrightarrow \overset{V \text{ arbitrary}}{\dot{w} + \nabla \cdot \mathbf{S} = -\mathbf{J}_f \cdot \mathbf{E}}, \quad \frac{d}{dt} (W_V^{\text{mech}} + W_V^{\text{field}}) = - \oint_{\partial V} \mathbf{S} \cdot d\mathbf{a}.$$

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Wave propagation in attenuating media

Specifically: homogeneous isotropic conductors, linear media.

Electric field drives the free currents:

Ohm's law $\mathbf{J}_f = \sigma \mathbf{E}$, σ : conductivity of the material.

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \dot{\mathbf{D}}.$$

$$\hookrightarrow \dot{\rho}_f = -\frac{\sigma}{\epsilon_0 \epsilon} \rho_f, \quad \rho_f(\mathbf{r}, t) = \rho_f(\mathbf{r}, t_0) \exp\left(-\frac{\sigma}{\epsilon_0 \epsilon} (t - t_0)\right),$$

assume $\rho_f(\mathbf{r}, t_0) = 0 \rightsquigarrow \rho_f(\mathbf{r}, t) = 0 \quad \forall t.$

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}}.$$

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Electromagnetic energy, frequency domain

Lossless uncharged nondispersive (. . .) linear media:

$$w = \frac{1}{2}(\epsilon_0 \mathbf{E} \cdot \hat{\epsilon} \mathbf{E} + \mu_0 \mathbf{H} \cdot \hat{\mu} \mathbf{H}), \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}, \quad \dot{w} + \nabla \cdot \mathbf{S} = 0,$$

$$\mathbf{E}(\mathbf{r}, t) = \text{Re } \tilde{\mathbf{E}}(\mathbf{r}) e^{i\omega t}, \quad \mathbf{H}(\mathbf{r}, t) = \text{Re } \tilde{\mathbf{H}}(\mathbf{r}) e^{i\omega t}$$

$\hookrightarrow \mathbf{S}, w$ oscillate in time.

Consider time-averaged quantities: $\bar{f}(t) = \frac{1}{T} \int_t^{t+T} f(t') dt'$ (FD) (exercise)

$$\hookrightarrow \bar{w} = \frac{1}{4} \text{Re} \left(\epsilon_0 \tilde{\mathbf{E}}^* \cdot \hat{\epsilon} \tilde{\mathbf{E}} + \mu_0 \tilde{\mathbf{H}}^* \cdot \hat{\mu} \tilde{\mathbf{H}} \right), \quad \bar{\mathbf{S}} = \frac{1}{2} \text{Re} \left(\tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}} \right).$$

$$\bar{\dot{w}} = \dot{\bar{w}} = 0, \quad \overline{\nabla \cdot \mathbf{S}} = \nabla \cdot \bar{\mathbf{S}} \rightsquigarrow \nabla \cdot \bar{\mathbf{S}} = 0, \quad \oint_V \bar{\mathbf{S}} \cdot d\mathbf{a} = 0;$$

“power balance”, conservation of energy.

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Telegrapher equation

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}}$$

$$\hookrightarrow \Delta \mathbf{E} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{E}} - \mu_0 \mu \sigma \dot{\mathbf{E}} = 0, \quad \Delta \mathbf{H} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{H}} - \mu_0 \mu \sigma \dot{\mathbf{H}} = 0,$$

Telegrapher equation.

Frequency domain: $\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}}(\mathbf{r}) e^{i\omega t}$,

$$\hookrightarrow \Delta \tilde{\mathbf{E}} + \left(\frac{\omega^2}{c^2} \epsilon \mu - i\omega \mu_0 \mu \sigma \right) \tilde{\mathbf{E}} = 0.$$

Nonconducting media $\sigma = 0$, $\Delta \tilde{\mathbf{E}} + \left(\frac{\omega^2}{c^2} \epsilon \mu \right) \tilde{\mathbf{E}} = 0.$

Define $\bar{\epsilon}$ such that $\frac{\omega^2}{c^2} \bar{\epsilon} \mu = \frac{\omega^2}{c^2} \epsilon \mu - i\omega \mu_0 \mu \sigma$, i.e. $\bar{\epsilon} = \epsilon - i \frac{\sigma}{\epsilon_0 \omega}$

$$\hookrightarrow \Delta \tilde{\mathbf{E}} + k^2 \bar{\epsilon} \mu \tilde{\mathbf{E}} = 0, \quad \text{Helmholtz equation, } \bar{\epsilon} \in \mathbb{C}, \quad k = \frac{\omega}{c}.$$

For given σ , the choice of the FD time dependence $\sim e^{\pm i\omega t}$ determines the sign of $\text{Im } \bar{\epsilon}$. (!)

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Wave attenuation

$$\Delta \tilde{\mathbf{E}} + k^2 \bar{\epsilon} \mu \tilde{\mathbf{E}} = 0, \quad \bar{\epsilon} \in \mathbb{C} \quad (\text{FD, } \exp(i\omega t), \omega > 0)$$

$$\hookrightarrow \text{solutions} \sim e^{i(\omega t - k\bar{n}z)} \quad \text{and} \sim e^{i(\omega t + k\bar{n}z)}$$

$$\text{with refractive index } \bar{n} = n' - in'' = \pm \sqrt{\bar{\epsilon} \mu} \in \mathbb{C}, \quad (!)$$

$$e^{-i(k\bar{n}z - \omega t)} = e^{-i(kn'z - \omega t)} e^{-kn''z},$$

damped plane wave solutions

$$\text{for } n' > 0, n'' > 0. \\ (n'' > 0: e^{-i(kn'z - \omega t)} \text{ is a forward traveling wave.})$$

Issues:

- penetration depth,
- S and w decay with z ,
- still transverse waves,
- E, H no longer in phase,
- notions of wavenumber, wavelength, phase velocity $\in \mathbb{C}$.

$$(\bar{\epsilon} \mu = \bar{n}^2 = (n')^2 - (n'')^2 - i2n'n'') \\ (\text{Modelling of gain: reverse the signs of } n'', \text{Im } \bar{\epsilon}.) \\ (\text{Choice of } e^{\pm i\omega t} \leftrightarrow \text{signs of } n'', \text{Im } \bar{\epsilon} \text{ indicate loss/gain.})$$

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Simulations in integrated optics

A typical setting:

- “uncharged dielectric medium”: $\nabla_{\mathbf{r}} \cdot \mathbf{J}_{\mathbf{r}} = 0$
- “linear medium”: $\mathbf{D} = \epsilon_0 \hat{\epsilon} \mathbf{E}$, $\mathbf{B} = \mu_0 \hat{\mu} \mathbf{H}$.
- “isotropic medium”: $\hat{\epsilon} = \epsilon \hat{1}$, $\hat{\mu} = \mu \hat{1}$.
- “nonmagnetic medium”: $\hat{\mu} = \hat{1}$.
- “lossless medium”: $\hat{\epsilon}^\dagger = \hat{\epsilon}$, $\hat{\mu}^\dagger = \hat{\mu}$, $(\epsilon, \mu \in \mathbb{R})$.
- “piecewise constant” \rightarrow “dependent on position”.
- “electric and magnetic field”: eliminate \mathbf{D} and \mathbf{B} , retain \mathbf{E} and \mathbf{H} .
- “governed by the curl equations”: divergence eqns. are satisfied.
- “frequency domain, time harmonic fields, frequency, wavelength”:
... as discussed.

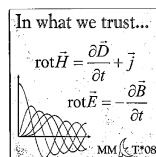
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Upcoming

Next lectures:

- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.
- Examples for dielectric optical waveguides.



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