

# *Optical Waveguide Theory (J)*



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### Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
  - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
  - Oblique semi-guided waves: 2-D integrated optics.
  - Summary, concluding remarks.

# *A touch of photonic crystals*

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“Photonic crystals”: ?

Keywords:

- A branch of photonics.
- Optics involving structures with (1-D, 2-D, 3-D) *spatial periodicity*.
- 1-D periodicity: Multilayer stacks / coatings, gratings, corrugated waveguides.
- 2-D periodicity: Corrugated dielectric slabs, membranes, gratings.
- 3-D periodicity: Bulk photonic crystals.
- “Molding the flow of light”  $\longleftrightarrow$  tunability, degrees of freedom in design.
- Defect cavities & defect waveguides in photonic crystals.
- Phenomena & fundamental research.
- Photonic crystal fibers.

Context of this lecture:

- Problems of general classical electromagnetics & methods as discussed; different emphasis.
- Periodicity: *Restrict computations to unit cells.*

## Structures with spatial periodicity

$\sim \exp(i\omega t)$  (FD)

Infinite system with periodic permittivity:

$$\epsilon(\mathbf{r} + \mathbf{g}) = \epsilon(\mathbf{r}) \quad \text{for all lattice vectors } \mathbf{g}.$$

↪ Consider Floquet-Bloch waves

(Floquet: 1-D, context of mechanics;  
Bloch: context of solid state physics.)

$$\begin{pmatrix} E \\ H \end{pmatrix}(\mathbf{r}) = U_{\mathbf{k}}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}},$$

$\mathbf{k}$ : wavevector of the FB wave,

$U_{\mathbf{k}}$ : a periodic function,  $U_{\mathbf{k}}(\mathbf{r} + \mathbf{g}) = U_{\mathbf{k}}(\mathbf{r})$ .

(A plane wave, modulated by a periodic function.)

{FB waves}: A **complete** basis for the periodic system.

(Bloch theorem: any solution can be written as a superposition of FB waves.)

(Background: Hilbert space theory, self-adjoint operators; familiar from Quantum theory.)

(Hermitian Hamiltonian and translation operators commute; Bloch waves are a simultaneous eigenbasis of these operators.)


(Required: Hermitian “Hamiltonian”  $\leftrightarrow$  Hermitian  $\hat{\epsilon}$ .)

( $U_{\mathbf{k}} = ?$ , but  $U_{\mathbf{k}}$  satisfies different equations than  $E, H \dots$ )

## Structures with spatial periodicity

$\mathbf{g}$ : a lattice vector, such that  $\epsilon(\mathbf{r} + \mathbf{g}) = \epsilon(\mathbf{r})$

$\sim \exp(i\omega t)$  (FD)

 
$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(\mathbf{r} + \mathbf{g}) = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{g}}. \quad (\text{QPBC})$$

(... if  $\mathbf{g}$  connects the boundaries of a unit cell.)


### FB-wave eigenproblem:

Given a wavevector  $\mathbf{k}$ , look for frequencies  $\omega \in \mathbb{R}$ , such that there exist nonzero solutions  $(\mathbf{E}, \mathbf{H})$  on a **unit cell domain**, with quasi-periodic boundary conditions (QPBC).

## Structures with spatial periodicity

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### FB-wave eigenproblem:

Given a wavevector  $\mathbf{k}$ , look for frequencies  $\omega \in \mathbb{R}$ , such that there exist nonzero solutions  $(\mathbf{E}, \mathbf{H})$  on a **unit cell domain**, with quasi-periodic boundary conditions (QPBC).


- Outcome:

$\exists \omega$  with  $(\mathbf{E}, \mathbf{H}) \neq 0$ :  $(\mathbf{k}, \omega) \in$  a frequency **band**, or

$\nexists \omega$  with  $(\mathbf{E}, \mathbf{H}) \neq 0$ :  $\omega \in$  a **bandgap** region.

 “Bandstructure” calculations.

- QPBC for  $\mathbf{k}$  are the same as for  $\mathbf{k} + \mathbf{K}$ , if  $\mathbf{K} \cdot \mathbf{g} = m2\pi$ ,  $m \in \mathbb{Z}$ .

 Restrict  $\mathbf{k}$  to the **first Brillouin zone**.

(Exclude  $\mathbf{k} + \mathbf{K} \forall \mathbf{g}, m$ )


( $\mathbf{K}$ : A vector of the reciprocal lattice.)

## Structures with spatial periodicity

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$\mathbf{g}$ : a lattice vector, such that  $\epsilon(\mathbf{r} + \mathbf{g}) = \epsilon(\mathbf{r})$

$\sim \exp(i\omega t)$  (FD)

 
$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(\mathbf{r} + \mathbf{g}) = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{g}}. \quad \text{(QPBC)}$$

(... if  $\mathbf{g}$  connects the boundaries of a unit cell.)

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### FB-wave eigenproblem:

Given a wavevector  $\mathbf{k}$ , look for frequencies  $\omega \in \mathbb{R}$ , such that there exist nonzero solutions  $(\mathbf{E}, \mathbf{H})$  on a **unit cell domain**, with quasi-periodic boundary conditions (QPBC).

(Include this in the list of computational problems of lecture D.)

(Bandstructure calculations: Information on infinite periodic structures.)

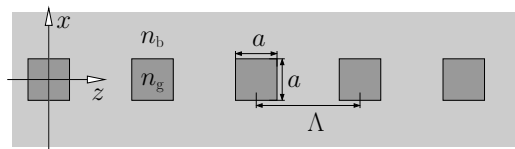
(Calculations on a (small) unit cell domain, typically computationally cheap.)

(Finite structures, (most) defects, external excitation, etc.: scattering solvers (FD, TD)

or resonance solvers required, on the full system domain.)

## A sequence of dielectric rods

$\sim \exp(i\omega t)$  (FD)



$$a = 0.4 \mu\text{m}, \Lambda = 1 \mu\text{m}, \\ n_b = 1.0, n_g = \sqrt{12}.$$

[Joannopoulos, Johnson, Winn, Meade, *Photonic Crystals: Molding the Flow of Light*, 2nd edition, Princeton, 2008.]

- 1-D periodicity,  $\epsilon(x, z) = \epsilon(x, z + \Lambda)$ .
- 2-D TE setting,  $E_y(x, z) = ?$ ,  $(\partial_x^2 + \partial_z^2 + k^2\epsilon)E_y = 0$ . (\*)
- Look for FB waves  $E_y(x, z) = u(x, z) e^{-i\beta z}$ .  
( $\beta$ : the FB wavenumber,  $u(x, z) = u(x, z + \Lambda) \forall z$ )
- $E_y(x, z + \Lambda) = u(x, z + \Lambda) e^{-i\beta(z + \Lambda)} = E_y(x, z) e^{-i\beta\Lambda}$   
↪ Restrict (\*) to  $z \in [0, \Lambda]$  with boundary conditions  
 $E_y(x, \Lambda) = e^{-i\beta\Lambda} E_y(x, 0)$ ,  $\partial_z E_y(x, \Lambda) = e^{-i\beta\Lambda} \partial_z E_y(x, 0)$ .
- Brillouin zone:  $K\Lambda = \pm m 2\pi \rightsquigarrow \beta \in [-\pi/\Lambda, \pi/\Lambda]$ .



(BEP simulations (Lecture G.24),  $\omega$  given,  $\beta$  determined from an eigenvalue problem.)  
 (Shaded region: above the “light line”,  $\omega^2 n_b^2 / c^2 > k_z^2$ , potentially leaky solutions.)





## ***Defect waveguides***

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(At a frequency in the bandgap of a photonic crystal:  $\exists$  “forbidden” regions  $\rightsquigarrow$  The waves travel elsewhere . . .)

Line defects in a square lattice of dielectric rods,  
excitation through conventional waveguides, 2-D QWEP simulations.

- A straight defect waveguide. 
- 90° corner in a defect waveguide. 

# *A touch of plasmonics*

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“Plasmonics”: ?

Keywords:

- A branch of photonics.
- Optics involving metals and metal surfaces.
- Interaction between the electromagnetic field and free electrons in the metal / at the surface.
- Strong field confinement, “beyond the diffraction limit”.
- “Strong” local fields, near field enhancement (nonlinearity).
- “Small” structures: Nano . . . .
- Applications: Sensing, focusing (“antennas”, microscopy), communication (short-range), chemistry, art.

Context of this lecture:

- Problems of general classical electromagnetics & methods as discussed; different emphasis.
- Presence of metals: complex (negative) permittivity, strong [dispersion](#), [losses](#); some concepts do not apply.
- Among the phenomena not encountered so far: [Surface plasmon polaritons](#) (SPPs).

## Surface plasmon polaritons

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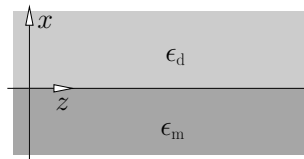
(Surface waves,

“plasmon”: oscillations of the free electron plasma,

“polariton”: strong interaction of the optical e.m. field with polarizable matter; here discussed merely as . . . )

### Optical waves confined at a metal / dielectric interface.

(. . . accepting the permittivities as given, disregarding any processes in the metal or dielectric that lead to this permittivity.)



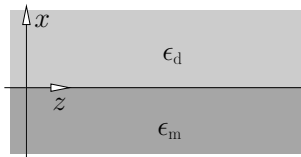
$x > 0$ : dielectric,  $\epsilon_d = n_d^2 \in \mathbb{R}$ .

$x < 0$ : metal,  $\epsilon_m \in \mathbb{C}$ .

(Coordinates in line with the previous discussion in this lecture, but different from literature “standard”.)

## Surface plasmon polaritons

$\sim \exp(i\omega t)$  (FD)



$x > 0$ : dielectric,  $\epsilon_d = n_d^2 \in \mathbb{R}$ .

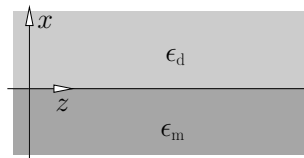
$x < 0$ : metal,  $\epsilon_m \in \mathbb{C}$ .

2-D TE/TM waves.

- Look for fields  $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix}(x) e^{-i\gamma z}$ ,  
 $\gamma = \beta - i\alpha \in \mathbb{C}$ ,  $\beta, \alpha \geq 0$ .
- Principal component  $\phi = \bar{E}_y$  (TE) and  $\phi = \bar{H}_y$  (TM),  
continuity of  $\phi$ ,  $\eta \partial_x \phi$  at the interface,  $\eta = 1$  (TE),  $\eta = 1/\epsilon$  (TM),  
$$\partial_x^2 \phi + (k^2 \epsilon - \gamma^2) \phi = 0 \quad \text{for } x < 0 \text{ and } x > 0.$$
- Ansatz:  
$$\phi(x) = \begin{cases} \phi_0 e^{-ik_d x}, & x > 0, & k_d = \chi_d - i\kappa_d, & \kappa_d > 0, \\ \phi_0 e^{ik_m x}, & x < 0, & k_m = \chi_m - i\kappa_m, & \kappa_m > 0. \end{cases}$$

## Surface plasmon polaritons

$\sim \exp(i\omega t)$  (FD)



$x > 0$ : dielectric,  $\epsilon_d = n_d^2 \in \mathbb{R}$ .

$x < 0$ : metal,  $\epsilon_m \in \mathbb{C}$ .

- $x > 0$ :  $k^2 \epsilon_d - k_d^2 - \gamma^2 = 0$ ,
- $x < 0$ :  $k^2 \epsilon_m - k_m^2 - \gamma^2 = 0$ .

- $x = 0$ : Continuity of  $\phi$ .

(Ansatz.)

$x = 0$ : Continuity of  $\eta \partial_x \phi \rightsquigarrow -k_d \eta_d = k_m \eta_m$ .

(TE):  $-k_d = k_m \rightsquigarrow$  No TE solution. (Required:  $\kappa_d > 0$  &  $\kappa_m > 0$ .)

$$\text{(TM): } -\frac{k_d}{\epsilon_d} = \frac{k_m}{\epsilon_m}.$$

(OK, if  $\text{Re } \epsilon_m < 0$ .)

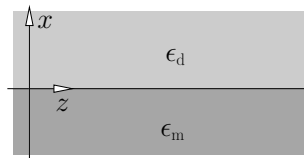
(No solution for an interface between pure dielectrics.)

$\curvearrowright$   $\gamma = \frac{\omega}{c} \sqrt{\frac{\epsilon_d \epsilon_m}{\epsilon_d + \epsilon_m}}$ , the dispersion equation for SPPs.

(Note that, in general,  $\epsilon_m(\omega)$ .)

## Surface plasmon polaritons

$\sim \exp(i\omega t)$  (FD)



$x > 0$ : dielectric,  $\epsilon_d = n_d^2 \in \mathbb{R}$ .

$x < 0$ : metal,  $\epsilon_m \in \mathbb{C}$ .

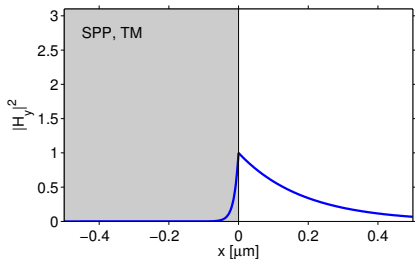
Characteristic lengths:

- $x > 0$ :  $|\phi(x)|^2 \sim e^{-2\kappa_d x} \rightsquigarrow d_d = \frac{1}{2\kappa_d}$ . (Penetration depth, dielectric.)
- $x < 0$ :  $|\phi(x)|^2 \sim e^{2\kappa_m x} \rightsquigarrow d_m = \frac{1}{2\kappa_m}$ . (Penetration depth, metal.)
- $|E_z|^2 \sim e^{-2\alpha z} \rightsquigarrow L_p = \frac{1}{2\alpha}$ , the SPP propagation length.

## Field profiles

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SPP, Ag/air,  $\lambda = 0.633 \mu\text{m}$ ,  
 $\epsilon_m = -14.5 - 1.2i$ ,  $\epsilon_d = 1.0$

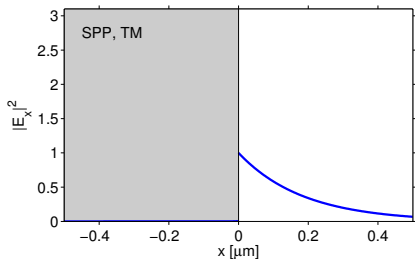


$L_p = 16 \mu\text{m}$ ,  
 $\beta/k = 1.036$ ,  
 $d_d = 190 \text{ nm}$ ,  
 $d_m = 12 \text{ nm}$ .

## Field profiles

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SPP, Ag/air,  $\lambda = 0.633 \mu\text{m}$ ,  
 $\epsilon_m = -14.5 - 1.2i$ ,  $\epsilon_d = 1.0$

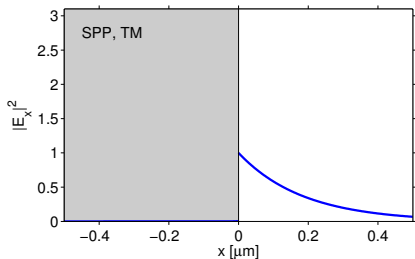


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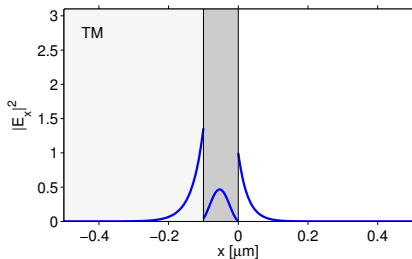
## Field profiles

SPP, Ag/air,  $\lambda = 0.633 \mu\text{m}$ ,  
 $\epsilon_m = -14.5 - 1.2i$ ,  $\epsilon_d = 1.0$



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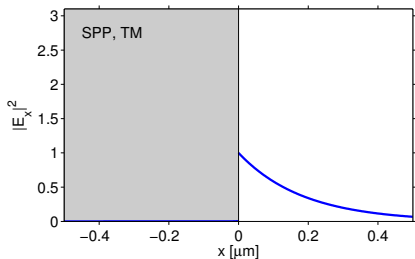
SiO<sub>2</sub>/Si(100 nm)/air,  $\lambda = 0.633 \mu\text{m}$ ,  
 $\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_p = \infty$ ,  
 $n_{\text{eff}} = 2.106$ ,  
 $d_{\text{air}} = 27 \text{ nm}$ .

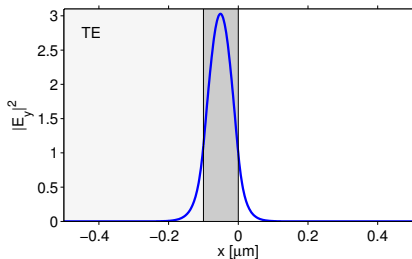
## Field profiles

SPP, Ag/air,  $\lambda = 0.633 \mu\text{m}$ ,  
 $\epsilon_m = -14.5 - 1.2i$ ,  $\epsilon_d = 1.0$



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SiO<sub>2</sub>/Si(100 nm)/air,  $\lambda = 0.633 \mu\text{m}$ ,  
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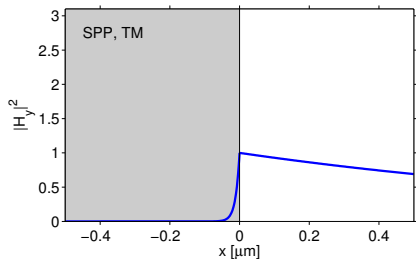


$L_p = \infty$ ,  
 $n_{\text{eff}} = 2.883$ ,  
 $d_{\text{air}} = 19 \text{ nm}$ .

## Field profiles

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SPP, Ag/air,  $\lambda = 1.550 \mu\text{m}$ ,  
 $\epsilon_m = -121 - 4.4i$ ,  $\epsilon_d = 1.0$

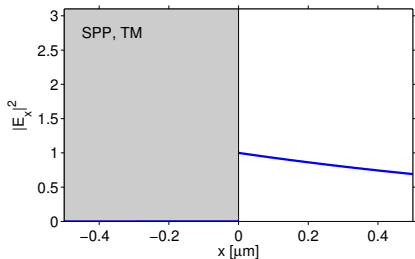


$L_p = 812 \mu\text{m}$ ,  
 $\beta/k = 1.0042$ ,  
 $d_d = 1350 \text{ nm}$ ,  
 $d_m = 11 \text{ nm}$ .

## Field profiles

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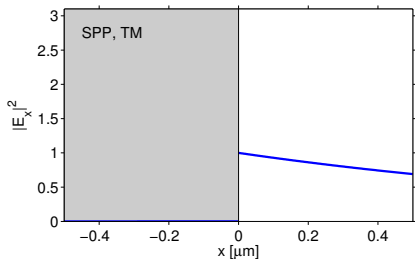
SPP, Ag/air,  $\lambda = 1.550 \mu\text{m}$ ,  
 $\epsilon_m = -121 - 4.4i$ ,  $\epsilon_d = 1.0$



$L_p = 812 \mu\text{m}$ ,  
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 $d_d = 1350 \text{ nm}$ ,  
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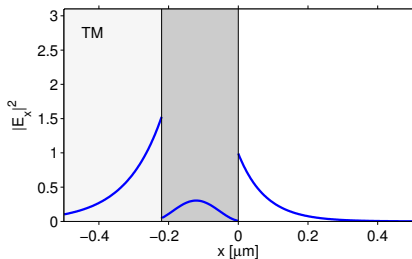
## Field profiles

SPP, Ag/air,  $\lambda = 1.550 \mu\text{m}$ ,  
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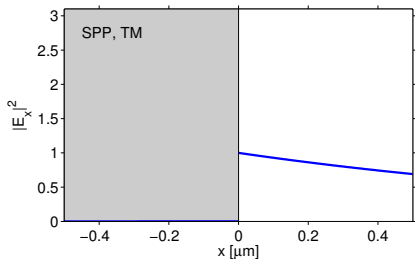
SiO<sub>2</sub>/Si(220 nm)/air,  $\lambda = 1.550 \mu\text{m}$ ,  
 $\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_p = \infty$ ,  
 $n_{\text{eff}} = 1.874$ ,  
 $d_{\text{air}} = 78 \text{ nm}$ .

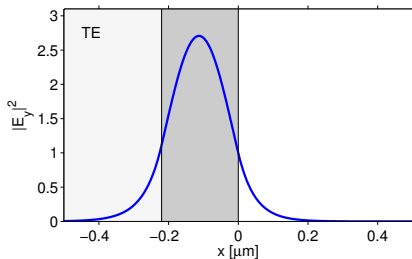
## Field profiles

SPP, Ag/air,  $\lambda = 1.550 \mu\text{m}$ ,  
 $\epsilon_m = -121 - 4.4i$ ,  $\epsilon_d = 1.0$



$L_p = 812 \mu\text{m}$ ,  
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 $d_d = 1350 \text{ nm}$ ,  
 $d_m = 11 \text{ nm}$ .

$\text{SiO}_2/\text{Si}(220 \text{ nm})/\text{air}$ ,  $\lambda = 1.550 \mu\text{m}$ ,  
 $\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_p = \infty$ ,  
 $n_{\text{eff}} = 2.805$ ,  
 $d_{\text{air}} = 47 \text{ nm}$ .

## Upcoming

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Next lectures:

- Oblique semi-guided waves: 2-D integrated optics.
- Summary, concluding remarks.

