

Optical Waveguide Theory (H)



Manfred Hammer*

Theoretical Electrical Engineering
Paderborn University, Paderborn, Germany

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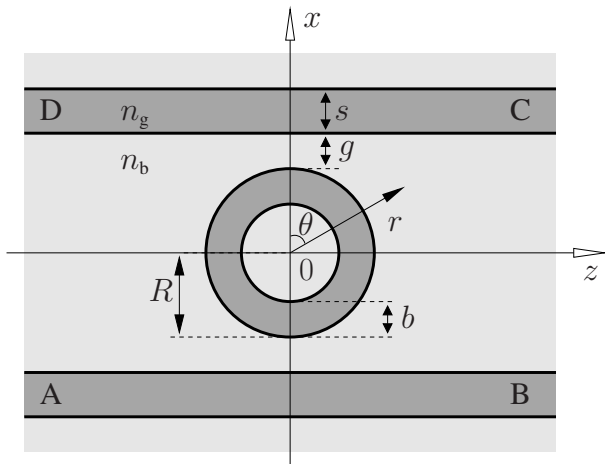
** Theoretical Electrical Engineering, Paderborn University
Warburger Straße 100, 33098 Paderborn, Germany*

*Phone: +49(0)5251/60-3560
E-mail: manfred.hammer@uni-paderborn.de*

Optical waveguide theory

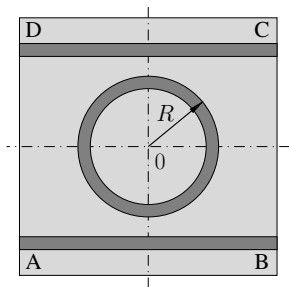
- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Hybrid analytical / numerical coupled mode theory.
 - Oblique semi-guided waves: 2-D integrated optics.

Circular traveling wave resonators

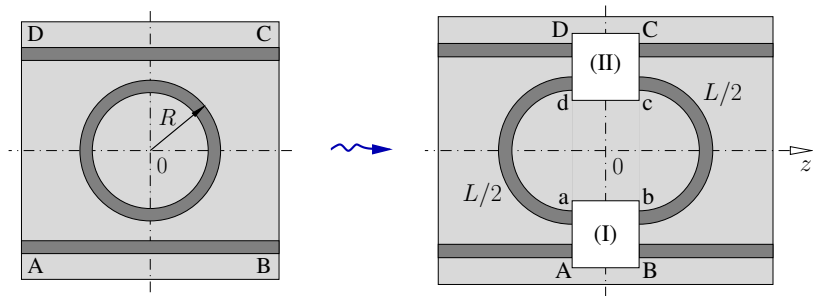


Integrated optical **micro-ring** or **micro-disk** resonators.

Ringresonator: Abstract model

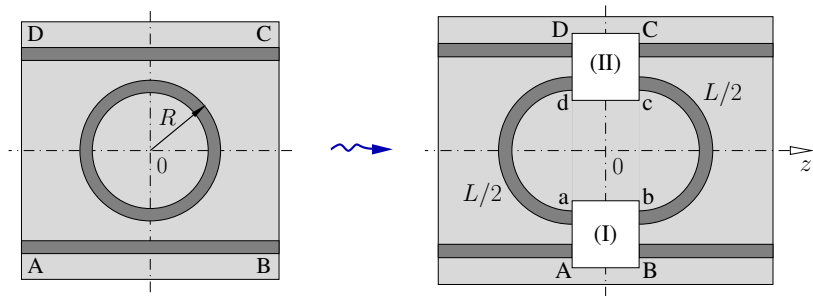


Ringresonator: Abstract model



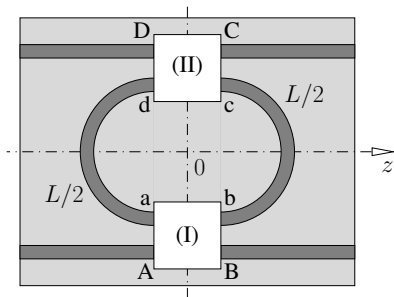
- Ringresonator \approx 2 couplers + 2 cavity segments

Ringresonator: Abstract model



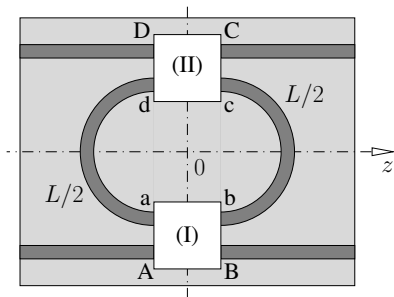
- Ringresonator \approx 2 couplers + 2 cavity segments
- CW description: $\mathbf{E}, \mathbf{H} \sim e^{i\omega t}$, $\omega = kc$, $k = 2\pi/\lambda$.

Couplers: Scattering matrices



- Uniform polarization, single mode waveguides.
- Linear, nonmagnetic (attenuating) elements.
- Backreflections are negligible.
- Interaction restricted to the couplers \leftrightarrow “port” definition.

Couplers: Scattering matrices



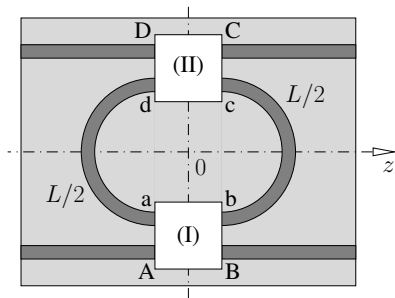
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↪ Symmetric coupler scattering matrices :

$$\begin{pmatrix} A_- \\ a_- \\ B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & \rho & \kappa \\ 0 & 0 & \chi & \tau \\ \rho & \chi & 0 & 0 \\ \kappa & \tau & 0 & 0 \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \\ B_- \\ b_- \end{pmatrix}$$

A_{\pm} , B_{\pm} , a_{\pm} , b_{\pm} : Amplitudes of waves traveling in $\pm z$ -direction.

Coupler symmetries

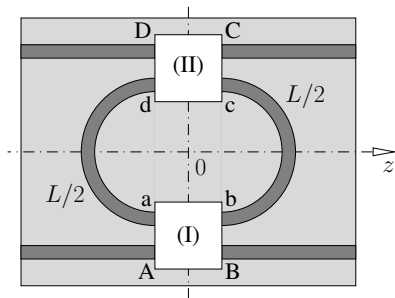


Symmetry $z \rightarrow -z$:

$$A_+ \rightarrow b_+ \stackrel{!}{=} B_- \rightarrow a_-$$

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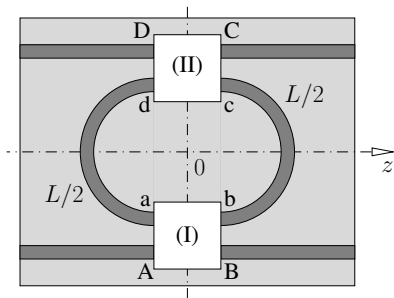


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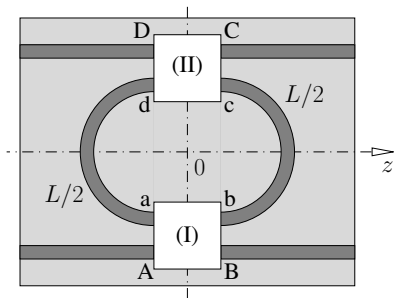
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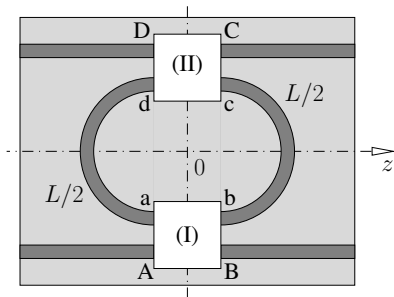
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Symmetry $x \rightarrow -x$, (I) = (II):

$$\curvearrowright \begin{pmatrix} D_- \\ d_- \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} C_- \\ c_- \end{pmatrix}, \quad \begin{pmatrix} C_+ \\ c_+ \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} D_+ \\ d_+ \end{pmatrix}.$$

Cavity segments



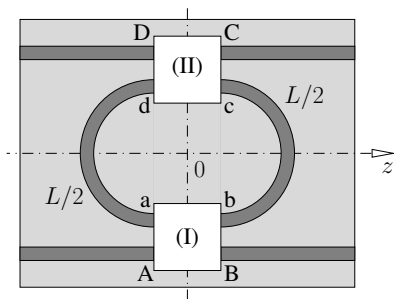
Field evolution $\sim e^{-i\gamma s}$
along the cavity core,
propagation distance s .

$$\gamma = \beta - i\alpha,$$

β : phase propagation constant,
 α : attenuation constant.

(\leftrightarrow bend modes, to come.)

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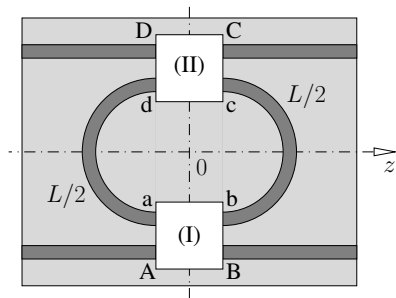
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Relations of amplitudes at the ends of the cavity segments :

$$\begin{aligned} c_- &= b_+ e^{-i\beta L/2} e^{-\alpha L/2}, & a_+ &= d_- e^{-i\beta L/2} e^{-\alpha L/2}, \\ b_- &= c_+ e^{-i\beta L/2} e^{-\alpha L/2}, & d_+ &= a_- e^{-i\beta L/2} e^{-\alpha L/2}. \end{aligned}$$

Output amplitudes

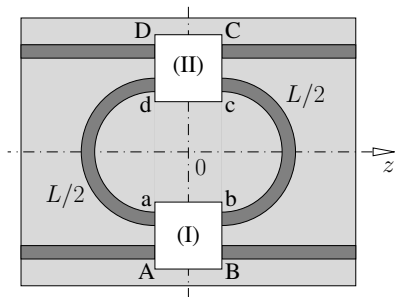


- Coupler scattering matrices
- + Cavity field evolution
- + External input amplitudes

$$A_+ = \sqrt{P_{\text{in}}},$$

$$B_- = C_- = D_+ = 0$$

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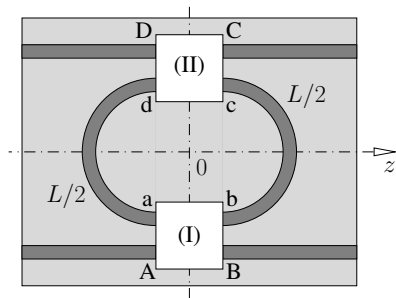
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External output amplitudes :

$$A_- = 0, \quad C_+ = 0, \quad D_- = \frac{\kappa^2 p}{1 - \tau^2 p^2} A_+, \quad B_+ = \left(\rho + \frac{\kappa^2 \tau p^2}{1 - \tau^2 p^2} \right) A_+,$$

$$p = e^{-i\beta L/2} e^{-\alpha L/2}.$$

Power transfer



Power drop: $P_D = |D_-|^2$,

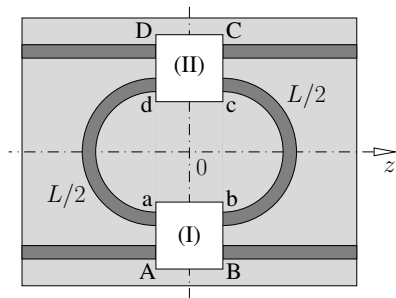
Transmission: $P_T = |B_+|^2$.

$$P_D = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L - 2\varphi)}$$

$$P_T = P_{\text{in}} \frac{|\rho|^2 (1 + |\tau|^2 d^2 e^{-2\alpha L} - 2|\tau| d e^{-\alpha L} \cos(\beta L - \varphi - \psi))}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L - 2\varphi)}$$

$$\tau =: |\tau| e^{i\varphi}, \quad d e^{i\psi} := \tau - \kappa^2 / \rho,$$

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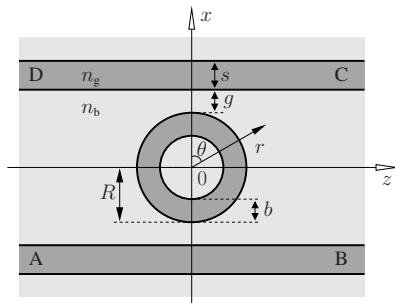
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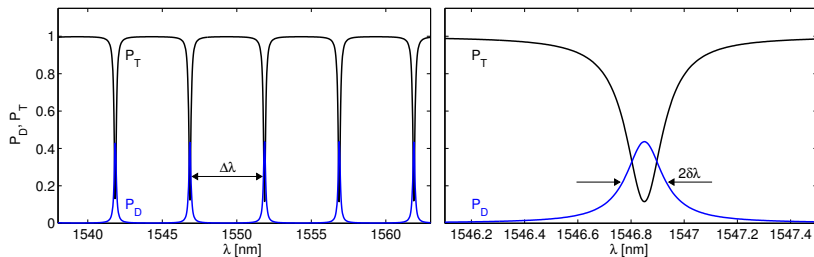
$$\tau =: |\tau| e^{i\varphi}, \quad d e^{i\psi} := \tau - \kappa^2/\rho, \quad L \neq 2\pi R.$$

Spectral response



$R = 50 \mu\text{m}$, $b = s = 1.0 \mu\text{m}$, $g = 0.9 \mu\text{m}$,
 $n_b = 1.45$, $n_g = 1.60$; 2-D, TE.

$\Delta\lambda = 5.0 \text{ nm}$, $2\delta\lambda = 0.17 \text{ nm}$,
 $F = 30$, $Q = 9400$, $P_{D,\text{res}} = 0.44$.



Resonances

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 $\beta L - 2\varphi = \beta L_{\text{cav}} - \phi$, $\phi = 2\beta l + 2\varphi$, $L_{\text{cav}} = 2\pi R$, $\partial_\lambda \phi \approx 0$.

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$$\beta = \frac{2m\pi + \phi}{L_{\text{cav}}} =: \beta_m \quad \text{integer } m; \quad P_D|_{\beta=\beta_m} = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{(1 - |\tau|^2 e^{-\alpha L})^2}.$$

Free spectral range

- Resonance next to β_m :

$$\beta_{m-1} = \frac{2(m-1)\pi + \phi}{L_{\text{cav}}} = \beta_m - \frac{2\pi}{L_{\text{cav}}}$$

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- $\partial_\lambda \beta = ?$

q_j : waveguide parameters with dimension length,

$$\beta(a\lambda, aq_j) = \beta(\lambda, q_j)/a, \quad \partial_a \Big|_{a=1}$$

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FSR :
$$\Delta\lambda = -\frac{2\pi}{L_{\text{cav}}} \left(\left. \frac{\partial\beta}{\partial\lambda} \right|_m \right)^{-1} \approx \frac{\lambda^2}{n_{\text{eff}} L_{\text{cav}}} \Big|_m, \quad n_{\text{eff}} = \beta/k.$$

(Free spectral range, the spectral distance (here: wavelength) between the drop peaks / the transmission dips.)

Spectral width of the resonances

- $$P_D = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L_{\text{cav}} - \phi)},$$
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- Expansion of cos-terms

$$\hookrightarrow \delta\beta = \pm \frac{1}{L_{\text{cav}}} \left(\frac{1}{|\tau|} e^{\alpha L/2} - |\tau| e^{-\alpha L/2} \right)$$

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$$\delta\beta = \pm \frac{1}{L_{\text{cav}}} \left(\frac{1}{|\tau|} e^{\alpha L/2} - |\tau| e^{-\alpha L/2} \right) \approx -\frac{\beta_m}{\lambda} \delta\lambda$$

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FWHM:
$$2\delta\lambda = \frac{\lambda^2}{\pi L_{\text{cav}} n_{\text{eff}}} \Big|_m \left(\frac{1}{|\tau|} e^{\alpha L/2} - |\tau| e^{-\alpha L/2} \right).$$

(Full width at half maximum of the spectral drop peaks / the transmission dips (wavelength).)

Finesse & Q-factor

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or
$$Q = kR n_{\text{eff}} F \quad \text{for} \quad L_{\text{cav}} = 2\pi R .$$

Performance versus coupling strength & losses

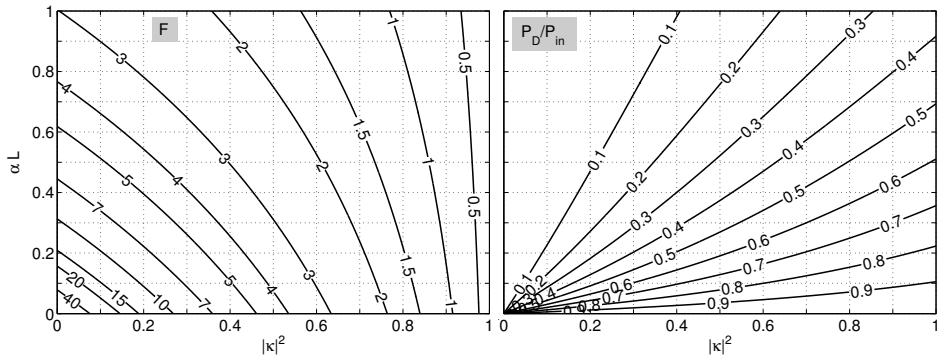
Assumption: Lossless coupler elements, $|\rho|^2 = |\tau|^2 = 1 - |\kappa|^2$.

$$F = \pi \frac{(\sqrt{1 - |\kappa|^2}) e^{-\alpha L/2}}{1 - (1 - |\kappa|^2) e^{-\alpha L}}, \quad P_{\text{D}}|_{\text{res}} = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{(1 - (1 - |\kappa|^2) e^{-\alpha L})^2}.$$

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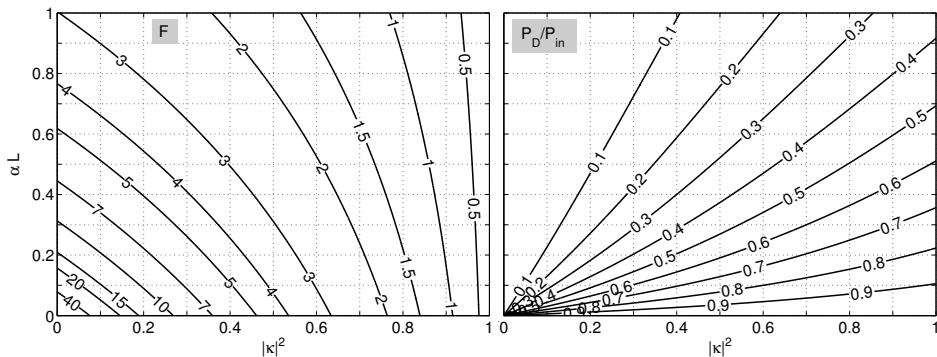
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Performance versus coupling strength & losses

Assumption: Lossless coupler elements, $|\rho|^2 = |\tau|^2 = 1 - |\kappa|^2$.

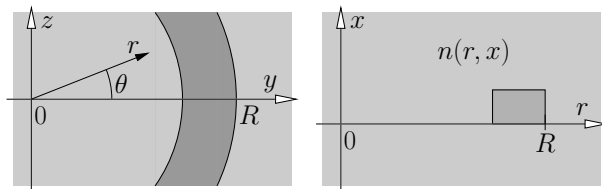
$$F = \pi \frac{(\sqrt{1 - |\kappa|^2}) e^{-\alpha L/2}}{1 - (1 - |\kappa|^2) e^{-\alpha L}}, \quad P_D|_{\text{res}} = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{(1 - (1 - |\kappa|^2) e^{-\alpha L})^2}.$$



$\alpha, \kappa = ?$

Modes of bent waveguides

$\sim \exp(i\omega t)$ (FD)



- Constant curvature \longleftrightarrow cylindrical coordinates r, θ, x .
- Bend radius R , $\partial_{\theta}\epsilon = 0$, $\partial_{\theta}n = 0$

$$\hookrightarrow \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (r, \theta, x) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (r, x) e^{-i\gamma R\theta}, \quad \text{bend modes,}$$

$\bar{\mathbf{E}}, \bar{\mathbf{H}}$: bend mode profile, components $\bar{E}_r, \bar{E}_{\theta}, \bar{E}_x, \bar{H}_r, \bar{H}_{\theta}, \bar{H}_x$,

$\gamma = \beta - i\alpha \in \mathbb{C}$: propagation constant,

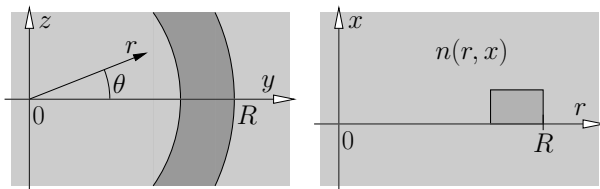
$\beta \in \mathbb{R}$: phase constant,

$\alpha \in \mathbb{R}$: attenuation constant.

(Exponent $i\gamma R\theta$: a convention, “propagation distance” $R\theta$.)

Modes of bent waveguides

$\sim \exp(i\omega t)$ (FD)



- Piecewise constant $n(r, x)$, $\psi \in \{\bar{E}_r, \bar{E}_\theta, \bar{E}_x, \bar{H}_r, \bar{H}_\theta, \bar{H}_x\}$,

$$\leftarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2} \right) \psi = 0, \quad \text{where } \partial n = 0,$$

& continuity conditions at interfaces (cylindrical coordinates),

& boundary conditions:

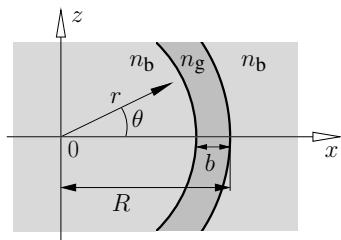
regularity at $r = 0$, outgoing waves at $r = \infty$, $x = \pm \infty$.

(or: normalizability versus x .)

Vectorial 3-D bend mode eigenvalue problem.

(Practical setting: computational domain $r_i < r < r_o$, $x_b < x < x_t$, PML boundary conditions / $\psi = 0$ at $r = r_i$.)

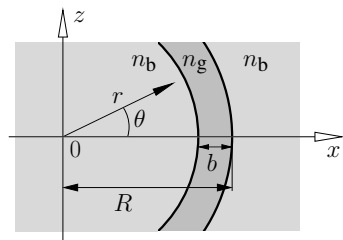
Modes of bent slab waveguides



$\sim \exp(i\omega t)$ (FD)

2-D TE/TM, cylind. coord. r, θ, y ,
 $\partial_y n = \partial_\theta n = 0$

Modes of bent slab waveguides



$\sim \exp(i\omega t)$ (FD)

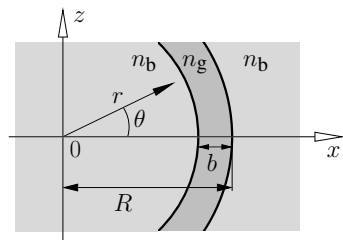
2-D TE/TM, cylind. coord. r, θ, y ,

$$\partial_y n = \partial_\theta n = 0$$

$$\left(\begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right) (r, \theta) = \left(\begin{array}{c} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{array} \right) (r) e^{-i\gamma R\theta},$$

bent slab mode $\{\bar{\mathbf{E}}, \bar{\mathbf{H}}, \gamma = \beta - i\alpha\}$.

Modes of bent slab waveguides



$\sim \exp(i\omega t)$ (FD)

2-D TE/TM, cylind. coord. r, θ, y ,

$$\partial_y n = \partial_\theta n = 0$$

$$\left(\begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right) (r, \theta) = \left(\begin{array}{c} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{array} \right) (r) e^{-i\gamma R\theta},$$

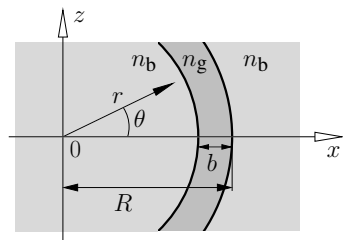
bent slab mode $\{\bar{\mathbf{E}}, \bar{\mathbf{H}}, \gamma = \beta - i\alpha\}$.

- Piecewise constant $n(r)$, $\phi = \bar{E}_y$ (TE), $\phi = \bar{H}_y$ (TM)

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2} \right) \phi = 0,$$

(Bessel differential equation with (complex) order γR .)

Modes of bent slab waveguides



$\sim \exp(i\omega t)$ (FD)

2-D TE/TM, cylind. coord. r, θ, y ,
 $\partial_y n = \partial_\theta n = 0$

$$\left(\begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right) (r, \theta) = \left(\begin{array}{c} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{array} \right) (r) e^{-i\gamma R\theta},$$

bent slab mode $\{\bar{\mathbf{E}}, \bar{\mathbf{H}}, \gamma = \beta - i\alpha\}$.

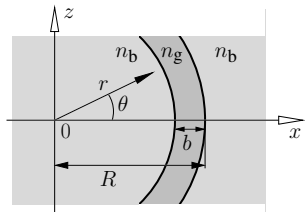
- Piecewise constant $n(r)$, $\phi = \bar{E}_y$ (TE), $\phi = \bar{H}_y$ (TM)

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2} \right) \phi = 0,$$

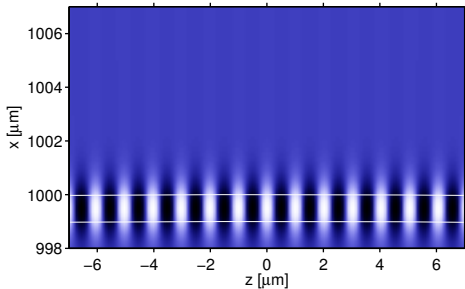
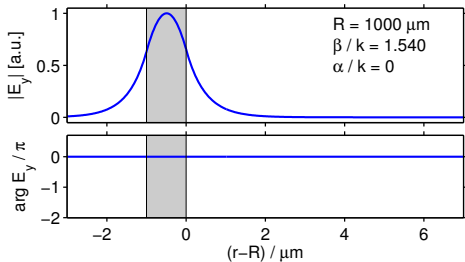
(Bessel differential equation with (complex) order γR .)

- Nonzero solutions,
- bounded at the origin, $\sim J_{\gamma R}(nkr)$ for $r < R - b$,
- outgoing exterior fields, $\sim H_{\gamma R}^{(2)}(nkr)$ for $r > R$, ($\sim \exp(i\omega t)$),
- continuity at interfaces: $\phi, \partial_r \phi$ (TE), $\phi, (\partial_r \phi)/n^2$ (TM).

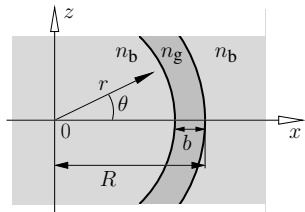
Bend modes, 2-D examples



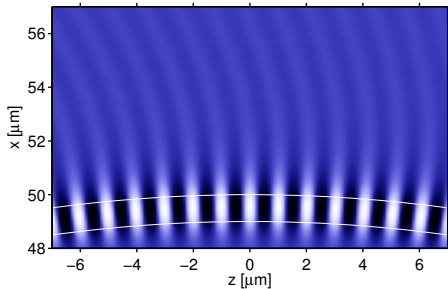
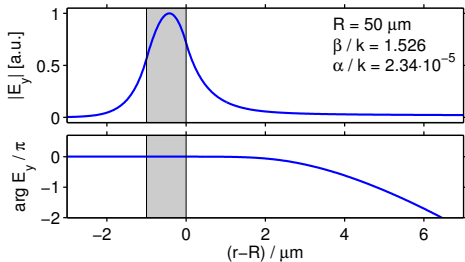
2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R = 1000 \mu\text{m}$.



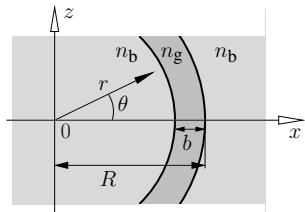
Bend modes, 2-D examples



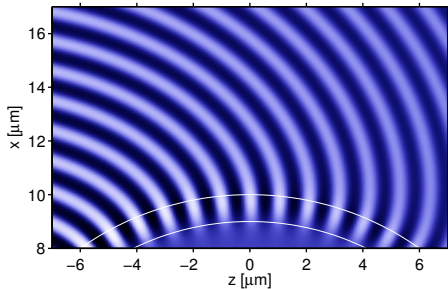
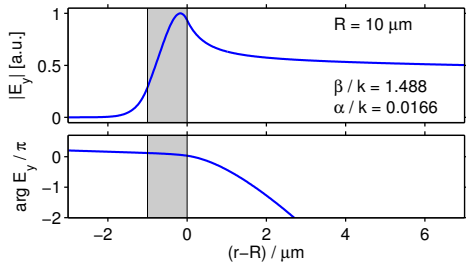
2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R = 50 \mu\text{m}$.



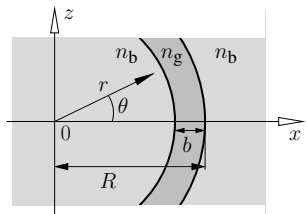
Bend modes, 2-D examples



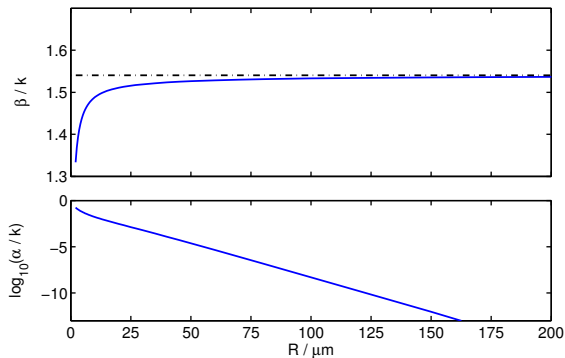
2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R = 10 \mu\text{m}$.



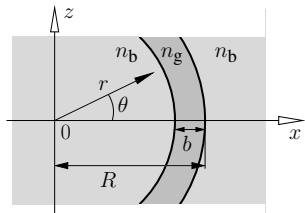
Propagation constant vs. bend radius



2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R \in [2, 200] \mu\text{m}$.



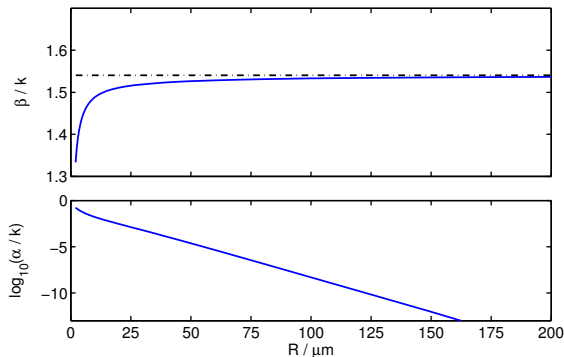
Propagation constant vs. bend radius



2-D, TE,

$n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,

$R \in [2, 200] \mu\text{m}$.



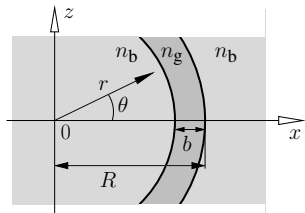
Alternative definition :

$$R' = R - b/2.$$

Identical physical fields

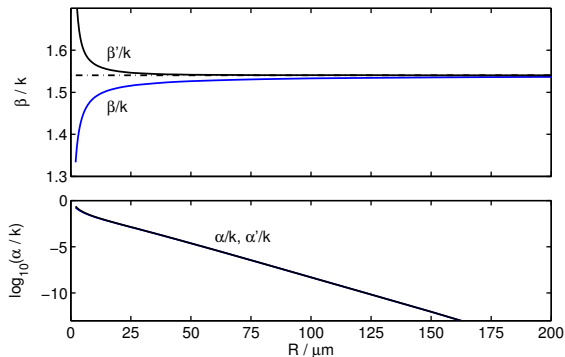
$$\begin{aligned} \gamma' R' &= \gamma R, \\ \gamma' &= \gamma \frac{R}{R - b/2}. \end{aligned}$$

Propagation constant vs. bend radius



2-D, TE,

$n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R \in [2, 200] \mu\text{m}$.



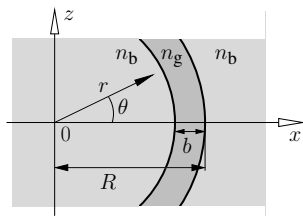
Alternative definition :
 $R' = R - b/2$.

Identical physical fields

$$\gamma' R' = \gamma R,$$

$$\gamma' = \gamma \frac{R}{R - b/2}.$$

Power & orthogonality



2-D TE/TM bend modes:

- Power flow: $S_r \neq 0$, $S_r, S_\theta \sim e^{-2\alpha R\theta}$, $S_\theta \sim |\phi|^2/r$

$$\int_0^\infty S_\theta(r) dr < \infty \quad \longleftrightarrow \quad \text{power normalization.}$$

- Orthogonality of nondegenerate bend modes, product

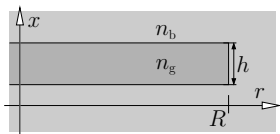
$$[\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2] = \int_0^\infty (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1) \cdot \mathbf{e}_\theta dr.$$

(Here [, ; ,] is complex valued.)

(Expressions $\sim \phi^2/r \longleftrightarrow$ convergence of the integrals.)

Bend modes supported by an angular disc segment

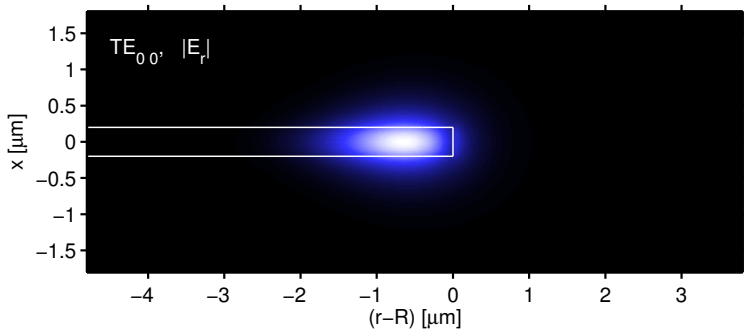
$$x \in [-3, 3] \mu\text{m},$$
$$(r - R) \in [-8, 4] \mu\text{m};$$



$$\lambda = 1.55 \mu\text{m},$$
$$n_b = 1.45,$$
$$n_g = 1.99,$$
$$h = 0.4 \mu\text{m},$$
$$R = 20 \mu\text{m};$$

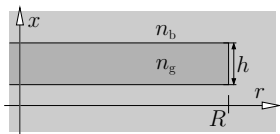
$$\text{TE}_{00}$$
$$\beta/k = 1.634$$
$$\alpha/k = 3.1 \cdot 10^{-8}$$

[JCMwave].



Bend modes supported by an angular disc segment

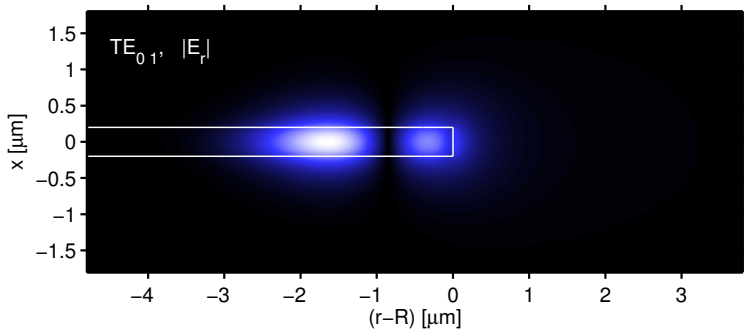
$$x \in [-3, 3] \mu\text{m},$$
$$(r - R) \in [-8, 4] \mu\text{m};$$



$$\lambda = 1.55 \mu\text{m},$$
$$n_b = 1.45,$$
$$n_g = 1.99,$$
$$h = 0.4 \mu\text{m},$$
$$R = 20 \mu\text{m};$$

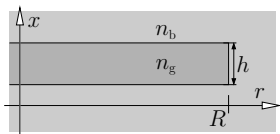
$$\text{TE}_{01}$$
$$\beta/k = 1.548$$
$$\alpha/k = 1.5 \cdot 10^{-5}$$

[JCMwave].



Bend modes supported by an angular disc segment

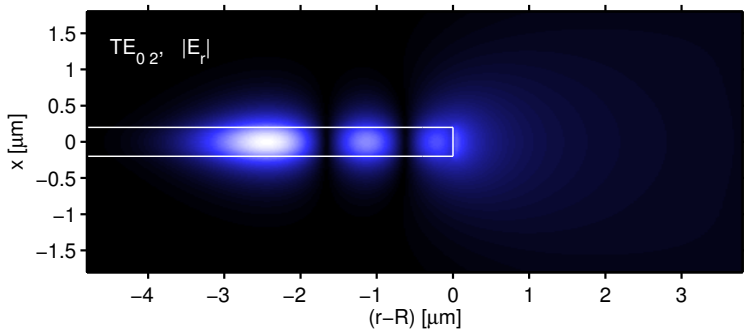
$$x \in [-3, 3] \mu\text{m},$$
$$(r - R) \in [-8, 4] \mu\text{m};$$



$$\lambda = 1.55 \mu\text{m},$$
$$n_b = 1.45,$$
$$n_g = 1.99,$$
$$h = 0.4 \mu\text{m},$$
$$R = 20 \mu\text{m};$$

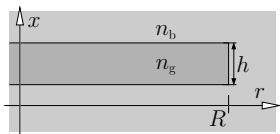
$$\text{TE}_{02}$$
$$\beta/k = 1.480$$
$$\alpha/k = 4.0 \cdot 10^{-4}$$

[JCMwave].



Bend modes supported by an angular disc segment

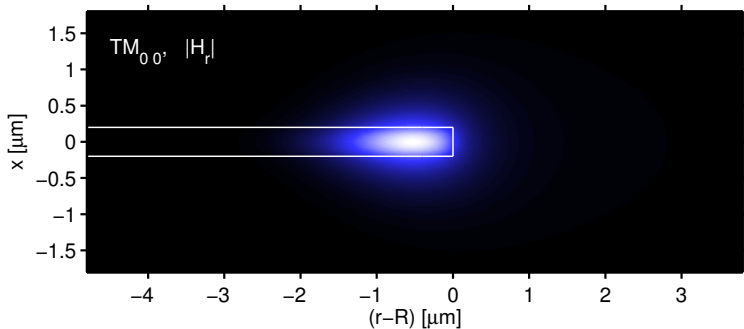
$$x \in [-3, 3] \mu\text{m},$$
$$(r - R) \in [-8, 4] \mu\text{m};$$



$$\lambda = 1.55 \mu\text{m},$$
$$n_b = 1.45,$$
$$n_g = 1.99,$$
$$h = 0.4 \mu\text{m},$$
$$R = 20 \mu\text{m};$$

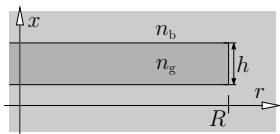
$$\text{TM}_{00}$$
$$\beta/k = 1.551$$
$$\alpha/k = 2.4 \cdot 10^{-5}$$

[JCMwave].



Bend modes supported by an angular disc segment

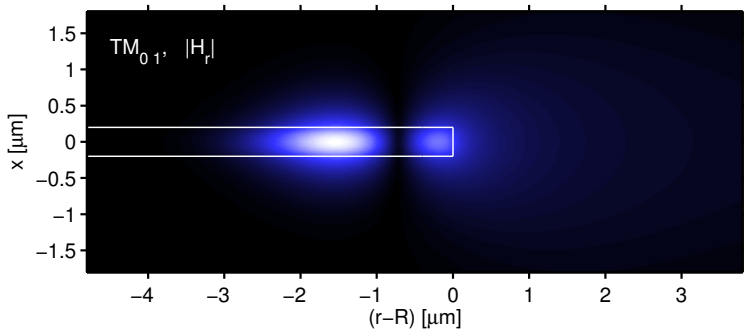
$$x \in [-3, 3] \mu\text{m},$$
$$(r - R) \in [-8, 4] \mu\text{m};$$



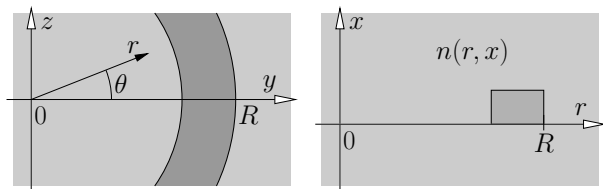
$$\lambda = 1.55 \mu\text{m},$$
$$n_b = 1.45,$$
$$n_g = 1.99,$$
$$h = 0.4 \mu\text{m},$$
$$R = 20 \mu\text{m};$$

$$\text{TM}_{01}$$
$$\beta/k = 1.468$$
$$\alpha/k = 7.6 \cdot 10^{-4}$$

[JCMwave].

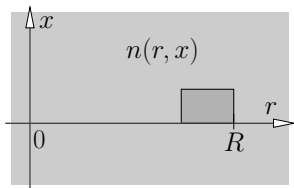
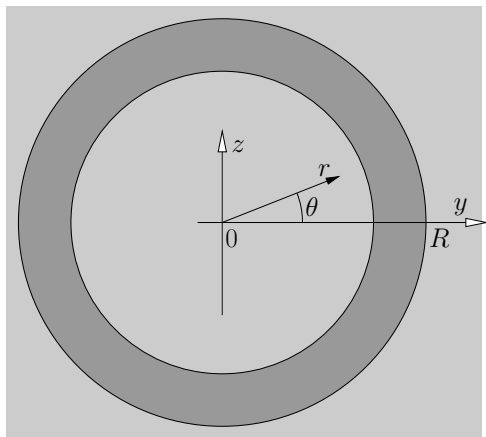


Circular microcavity



Bend modes

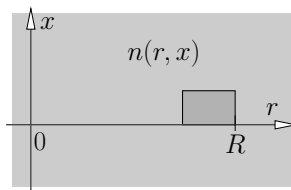
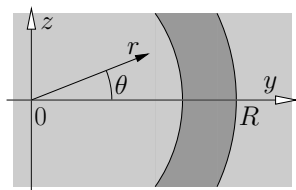
Circular microcavity



Bend modes \longleftrightarrow Whispering gallery resonances.

(Terms not always clearly distinguished.)

Whispering gallery resonances



(FD)

- Full cavity, $\theta \in [0, 2\pi]$:
Look for resonances in the form of **whispering gallery modes**

$$\left(\begin{matrix} \mathbf{E} \\ \mathbf{H} \end{matrix} \right) (r, \theta, x, t) = \left(\begin{matrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{matrix} \right) (r, x) e^{i\omega_c t - im\theta}, \quad \text{+c.c.}$$

Quasi-Normal-Modes, QNMs

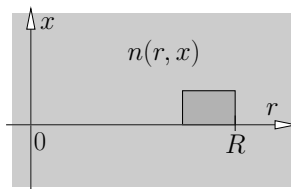
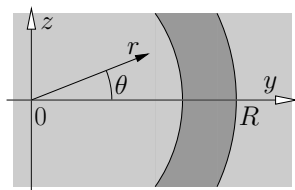
$\bar{\mathbf{E}}, \bar{\mathbf{H}}$: **WGM profile**, components $\tilde{E}_r, \tilde{E}_\theta, \tilde{E}_x, \tilde{H}_r, \tilde{H}_\theta, \tilde{H}_x$,

$m \in \mathbb{Z}$: **angular order**,

$\omega_c = \omega'_c + i\omega''_c \in \mathbb{C}$: **eigenfrequency**, $\omega'_c, \omega''_c \in \mathbb{R}$.

Q-factor $Q = \omega'_c / (2\omega''_c)$, resonance wavelength $\lambda_r = 2\pi c / \omega'_c$, outgoing radiation, FWHM: $2\delta\lambda = \lambda_r / Q$.

Whispering gallery resonances



(FD)

- Piecewise constant $n(r, x)$, $\psi \in \{\tilde{E}_r, \tilde{E}_\theta, \tilde{E}_x, \tilde{H}_r, \tilde{H}_\theta, \tilde{H}_x\}$, (Dispersion ?)

$$\leftarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \left(\frac{\omega_c^2}{c^2} n^2 - \frac{m^2}{r^2} \right) \psi = 0, \quad \text{where } \partial n = 0,$$

& continuity conditions at interfaces (cylindrical coordinates),

& boundary conditions:

regularity at $r = 0$, outgoing waves at $r = \infty$, $x = \pm \infty$.

(or: normalizability versus x .)

Vectorial eigenproblem for whispering gallery resonances.

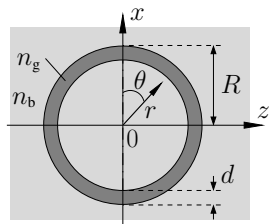
(Practical setting: computational domain $r_i < r < r_o$, $x_b < x < x_t$, PML boundary conditions / $\psi = 0$ at $r = r_i$.)

2-D whispering gallery resonances

... as discussed for the 2-D TE/TM bend modes.

(WGMs: Bessel differential equation of integer order.)
(Notation: $\text{WGM}(\rho, m)$ — mode of radial order ρ and angular order m .)

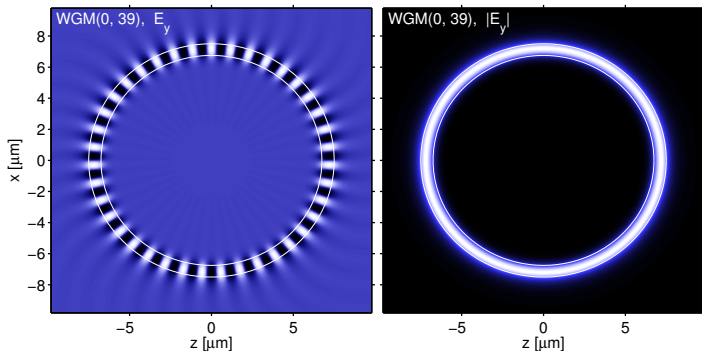
2-D whispering gallery resonances



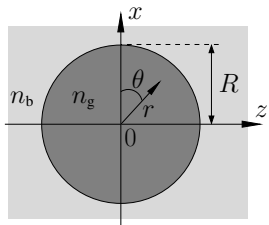
TE, $R = 7.5 \mu\text{m}$, $d = 0.75 \mu\text{m}$, $n_g = 1.5$, $n_b = 1.0$.

WGM(0, 39):

$\lambda_r = 1.5637 \mu\text{m}$, $Q = 1.1 \cdot 10^5$, $2\delta\lambda = 1.4 \cdot 10^{-5} \mu\text{m}$.



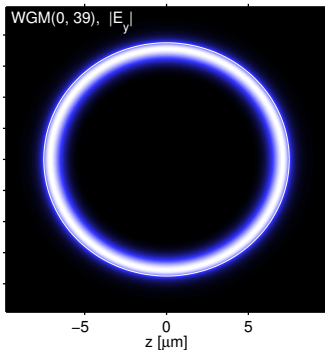
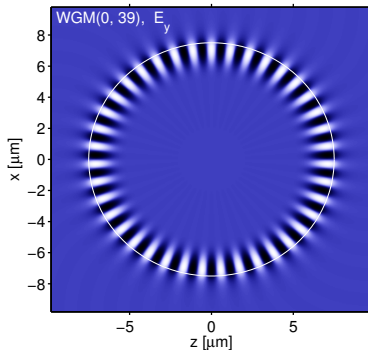
2-D whispering gallery resonances



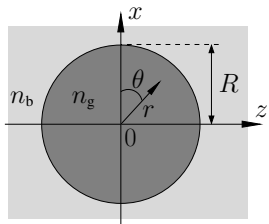
TE, $R = 7.5 \mu\text{m}$, $n_g = 1.5$, $n_b = 1.0$.

WGM(0, 39):

$\lambda_r = 1.6025 \mu\text{m}$, $Q = 5.7 \cdot 10^5$, $2\delta\lambda = 2.8 \cdot 10^{-6} \mu\text{m}$.



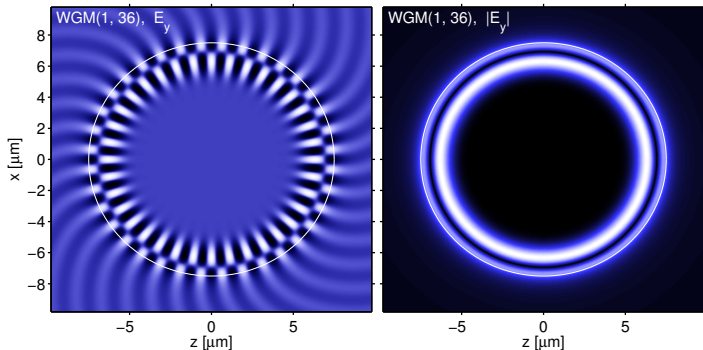
2-D whispering gallery resonances



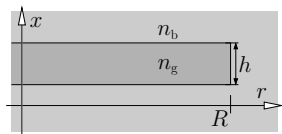
TE, $R = 7.5 \mu\text{m}$, $n_g = 1.5$, $n_b = 1.0$.

WGM(1, 36):

$\lambda_r = 1.5367 \mu\text{m}$, $Q = 2.2 \cdot 10^3$, $2\delta\lambda = 7.0 \cdot 10^{-4} \mu\text{m}$.



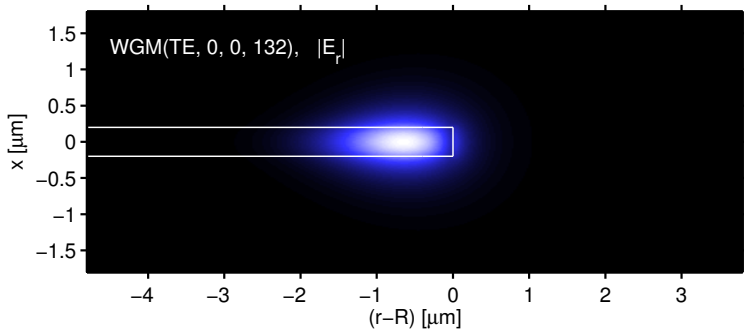
WGMs supported by a circular slab disc



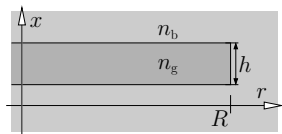
$$\begin{aligned}\lambda &= 1.55 \mu\text{m}, \\ n_b &= 1.45, \\ n_g &= 1.99, \\ h &= 0.4 \mu\text{m}, \\ R &= 20 \mu\text{m};\end{aligned}$$

$$\begin{aligned}x &\in [-3, 3] \mu\text{m}, \\ (r - R) &\in [-8, 4] \mu\text{m};\end{aligned}$$

$$\begin{aligned}\text{WGM}(\text{TE}, 0, 0, 132) \\ \lambda_r &= 1.555 \mu\text{m} \\ Q &= 6.9 \cdot 10^6 \\ [\text{JCMwave}].\end{aligned}$$



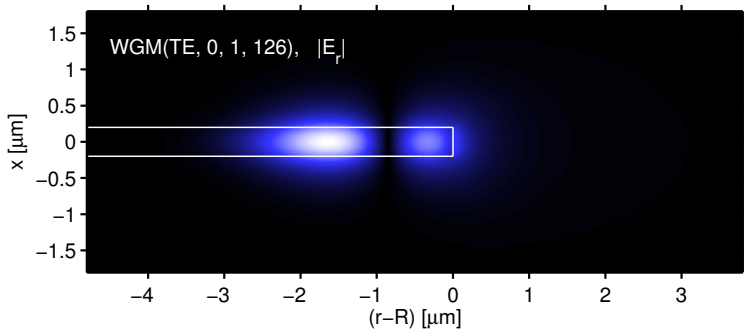
WGMs supported by a circular slab disc



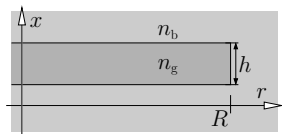
$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $h = 0.4 \mu\text{m}$,
 $R = 20 \mu\text{m}$;

$x \in [-3, 3] \mu\text{m}$,
 $(r - R) \in [-8, 4] \mu\text{m}$;

WGM(TE, 0, 1, 126)
 $\lambda_r = 1.545 \mu\text{m}$
 $Q = 1.7 \cdot 10^4$
[JCMwave].



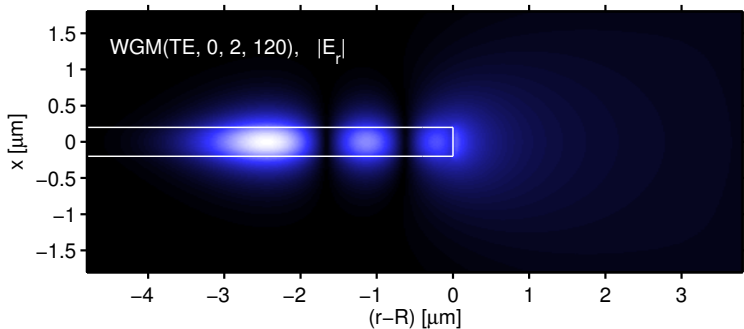
WGMs supported by a circular slab disc



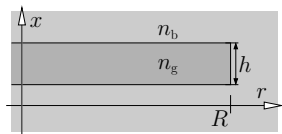
$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $h = 0.4 \mu\text{m}$,
 $R = 20 \mu\text{m}$;

$x \in [-3, 3] \mu\text{m}$,
 $(r - R) \in [-8, 4] \mu\text{m}$;

WGM(TE, 0, 2, 120)
 $\lambda_r = 1.550 \mu\text{m}$
 $Q = 5.7 \cdot 10^2$
[JCMwave].



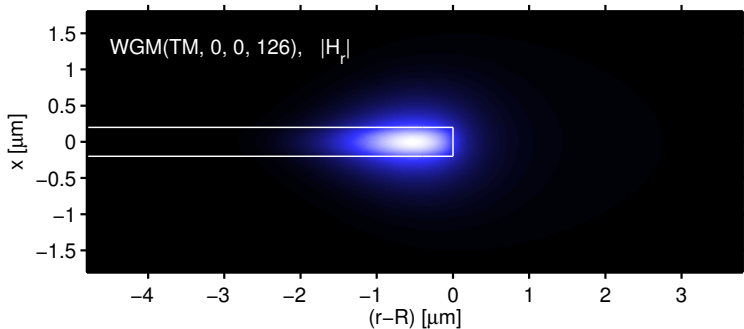
WGMs supported by a circular slab disc



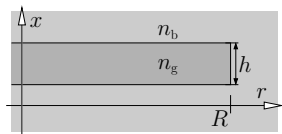
$$\begin{aligned}\lambda &= 1.55 \mu\text{m}, \\ n_b &= 1.45, \\ n_g &= 1.99, \\ h &= 0.4 \mu\text{m}, \\ R &= 20 \mu\text{m};\end{aligned}$$

$$\begin{aligned}x &\in [-3, 3] \mu\text{m}, \\ (r - R) &\in [-8, 4] \mu\text{m};\end{aligned}$$

$$\begin{aligned}\text{WGM(TM, 0, 0, 126)} \\ \lambda_r &= 1.547 \mu\text{m} \\ Q &= 1.0 \cdot 10^4 \\ [\text{JCMwave}].\end{aligned}$$



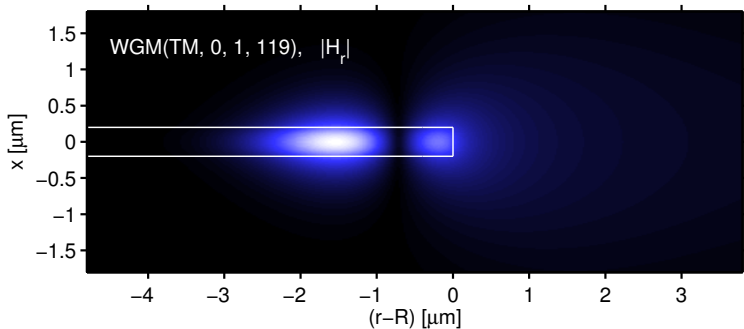
WGMs supported by a circular slab disc



$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $h = 0.4 \mu\text{m}$,
 $R = 20 \mu\text{m}$;

$x \in [-3, 3] \mu\text{m}$,
 $(r - R) \in [-8, 4] \mu\text{m}$;

WGM(TM, 0, 1, 119)
 $\lambda_r = 1.550 \mu\text{m}$
 $Q = 3.0 \cdot 10^2$
[JCMwave].



Bend modes versus whispering gallery resonances

(Field supported by a full circular cavity.)
(Incompatible models, in principle.)

[BWG] $\omega \in \mathbb{R}$ given, $\gamma = \beta - i\alpha \in \mathbb{C}$ eigenvalue,

$$\Phi(r, \theta, t) = \phi(r) e^{i\omega t - i\beta R\theta} e^{-\alpha R\theta}.$$

[WGM] $\omega_c = \omega_c + i\omega_c'' \in \mathbb{C}$ eigenvalue, $m \in \mathbb{Z}$ given,

$$\Psi(r, \theta, t) = \psi(r) e^{i\omega_c' t - im\theta} e^{-\omega_c'' t}.$$

Look at a resonant low-loss configuration:

- Translate $\omega \approx \omega_c'$, $m \approx \beta R$.
- Equate the power loss during one time period $T = 2\pi/\omega \approx 2\pi/\omega_c'$
 $\rightsquigarrow \beta/\alpha \approx \omega_c'/\omega_c'' = 2Q$.

Upcoming

Next lectures:

- Coupled mode theory, perturbation theory.
- A touch of photonic crystals; a touch of plasmonics.
- Hybrid analytical / numerical coupled mode theory.

