Optical Waveguide Theory (D)

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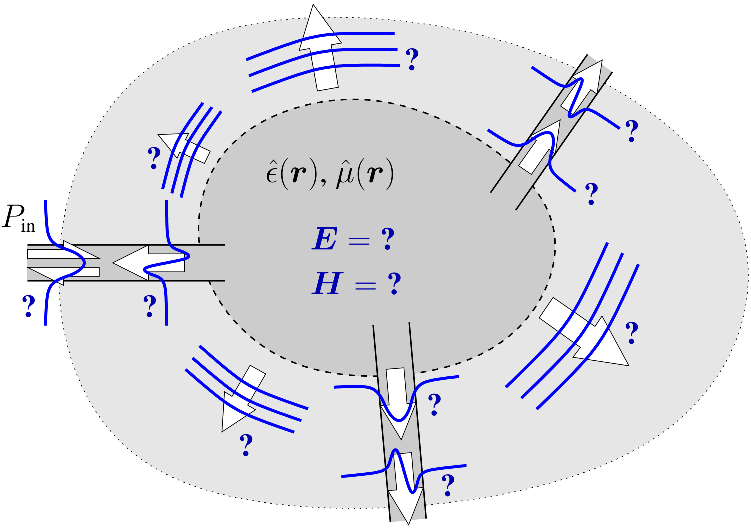
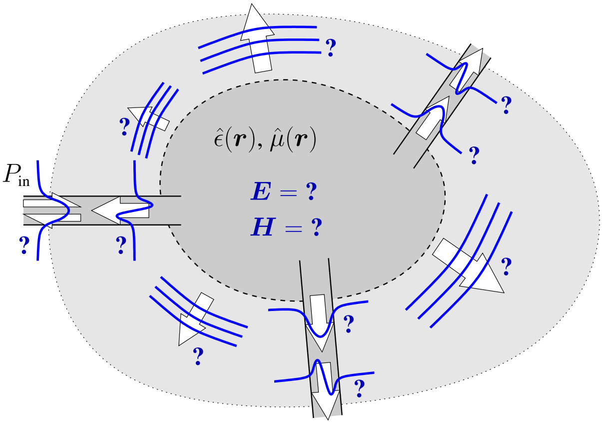
Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- J A touch of photonic crystals; a touch of plasmonics.
- Hybrid analytical / numerical coupled mode theory.
- Oblique semi-guided waves: 2-D integrated optics.

Guided wave scattering problems, schematically



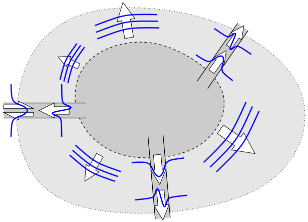
Guided wave scattering problems, schematically



Given $\hat{\epsilon}(\boldsymbol{r}),\ \hat{\mu}(\boldsymbol{r})$ & external excitation (incoming guided mode),
determine $\boldsymbol{E}, \boldsymbol{H}$ within the computational domain
& determine the optical power carried by outgoing waves.

Scattering problems, time domain

(TD)



$$\boldsymbol{E}(\boldsymbol{r},t),\;\; \boldsymbol{H}(\boldsymbol{r},t),$$
$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\mu_0\hat{\mu}\dot{\boldsymbol{H}},$$
$$\boldsymbol{\nabla}\times\boldsymbol{H} = \epsilon_0\hat{\epsilon}\dot{\boldsymbol{E}}.$$

Scattering problems, time domain

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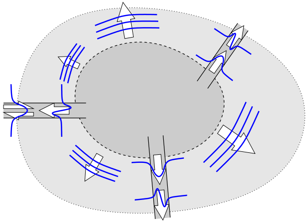

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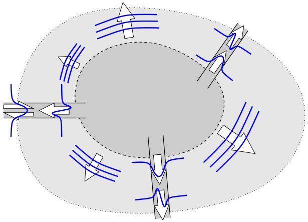
- $\begin{pmatrix} \text{3-D} \\ \text{2-D} \\ \text{1-D} \end{pmatrix}$ computational domain $\times$ time interval.
- Initial & boundary conditions $\longleftrightarrow$ incident waves.
- "Local" time-explicit iterative schemes possible (e.g. FDTD).
- Time evolution available; direct modeling of pulse propagation.
- Dispersion (. . . ?).
- Guided wave excitation (. . . ?).
- Fourier transform $\longrightarrow$ spectral information.

Scattering problems, frequency domain

(FD)



$$\boldsymbol{E}(\boldsymbol{r}),\ \boldsymbol{H}(\boldsymbol{r}),\ \sim \exp(\mathrm{i}\omega t),$$
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Scattering problems, frequency domain

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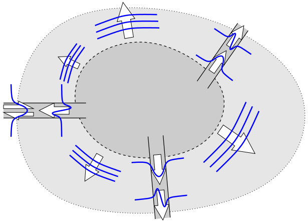

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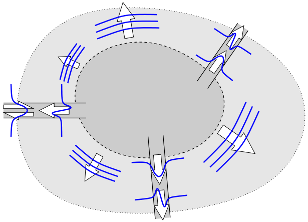
- $\begin{pmatrix} \text{3-D} \\ \text{2-D} \\ \text{1-D} \end{pmatrix}$ computational domain.
- "$\mathsf{M}(\overrightarrow{\text{field}}) = (\overrightarrow{\text{excitation}})$";
  matrix needs to be determined, stored; system needs to be solved.
- Spectral information directly available.
- Dispersion — straightforward.
- Guided wave excitation — straightforward.
- Fourier transform $\longrightarrow$ time evolution / pulse propagation.

Open problems

(TD & FD)



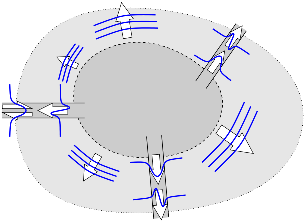
"Open" spatial computational domain
⇝ boundary conditions need to

- permit outgoing radiated fields
  & outgoing (reflected) guided modes to exit the domain,
- launch the incoming external excitation.

⇜⇝ simulate a nonexisting boundary, an unlimited domain.

Open problems

(TD & FD)



"Open" spatial computational domain

⟿ boundary conditions need to

- permit outgoing radiated fields
  & outgoing (reflected) guided modes to exit the domain,
- launch the incoming external excitation.

⬳ simulate a nonexisting boundary, an unlimited domain.

Keywords:
- transparent-influx boundary conditions,
- absorbing boundary conditions,
- perfectly matched layers (PMLs).

2-D problems

$$\hat{\epsilon} = \epsilon\hat{1},\ \hat{\mu} = \mu\hat{1},\ \sim \exp(\mathrm{i}\omega t) \qquad \text{(FD)}$$

$$\begin{pmatrix} \partial_y E_z - \partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0\mu \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \qquad \begin{pmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

2-D problems

$$\hat{\epsilon} = \epsilon \hat{1}, \;\; \hat{\mu} = \mu \hat{1}, \;\; \sim \exp(\mathrm{i}\omega t) \qquad \text{(FD)}$$

Assume $\partial_y \epsilon = 0$, $\partial_y \mu = 0$; consider solutions $\partial_y \boldsymbol{E} = 0$, $\partial_y \boldsymbol{H} = 0$:

$$\begin{pmatrix} -\partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y \end{pmatrix} = -\mathrm{i}\omega\mu_0\mu \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \qquad \begin{pmatrix} -\partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

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Two decoupled sets of equations:

- $\{E_y, H_x, H_z\}$: transverse electric (TE) fields, $\boldsymbol{E} \perp$ x-z-plane.
- $\{H_y, E_x, E_z\}$: transverse magnetic (TM) fields, $\boldsymbol{H} \perp$ x-z-plane.

(Different conventions on the use of TE, TM.)

(Applies also to the TD.)

2-D TE waves

$k^2 = \omega^2/c^2 = \omega^2\epsilon_0\mu_0$ (FD)

- Principal component E_y,

$$H_x = \frac{-\mathrm{i}}{\omega\mu_0\mu}\partial_z E_y, \quad H_z = \frac{\mathrm{i}}{\omega\mu_0\mu}\partial_x E_y, \quad \mathrm{i}\omega\epsilon_0\epsilon E_y = \partial_z H_x - \partial_x H_z$$

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$$\partial_x \frac{1}{\mu}\partial_x E_y + \partial_z \frac{1}{\mu}\partial_z E_y + k^2\epsilon E_y = 0. \qquad (*)$$

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- Continuity of E_y, $\frac{1}{\mu}\partial_n E_y$ required at interfaces with normal $\boldsymbol{n}$.

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- If $\mu = 1$: $\epsilon(x, z)$ (!)

$$\hookrightarrow \qquad \partial_x^2 E_y + \partial_z^2 E_y + k^2\epsilon E_y = 0, \qquad (**)$$

scalar 2-D (TE) Helmholtz equation (E_y, $\partial_n E_y$ continuous).

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(Reflection / transmission problems: s-polarized waves satisfy $(*)$, $(**)$.)

$k^2 = \omega^2/c^2 = \omega^2\epsilon_0\mu_0$ (FD)

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$$E_x = \frac{\mathrm{i}}{\omega\epsilon_0\epsilon}\partial_z H_y, \quad E_z = \frac{-\mathrm{i}}{\omega\epsilon_0\epsilon}\partial_x H_y, \quad -\mathrm{i}\omega\mu_0\mu H_y = \partial_z E_x - \partial_x E_z$$

2-D TM waves

$$k^2 = \omega^2/c^2 = \omega^2 \epsilon_0 \mu_0 \quad \text{(FD)}$$

- Principal component H_y,

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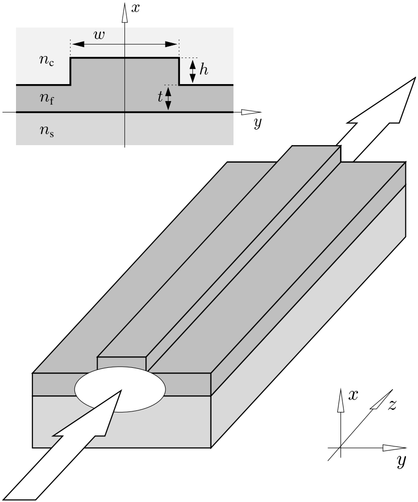
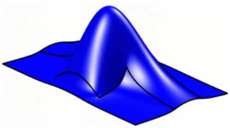
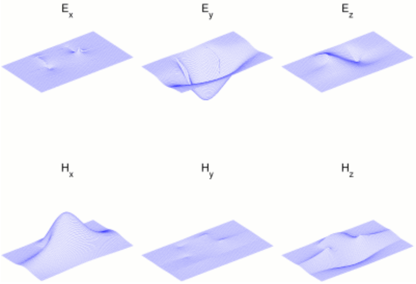
(Reflection / transmission problems: p-polarized waves satisfy $(*)$, $(**)$.)

Rib waveguide

. . . variant of an integrated optical waveguide with 2-D confinement







Waveguides: Mode problems

$$\nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\mu\boldsymbol{H}, \quad \nabla \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}. \qquad \sim \exp(\mathrm{i}\omega t) \qquad \text{(FD)}$$

- Waveguide: a system that is homogeneous along its axis z,
  $\partial_z\epsilon = 0,\ \partial_z\mu = 0,\ \partial_z n = 0.$

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- Look for solutions (modes) that vary harmonically with z:
  $\boldsymbol{E}(x, y, z) = \bar{\boldsymbol{E}}(x, y)\,\mathrm{e}^{-\mathrm{i}\beta z}, \quad \boldsymbol{H}(x, y, z) = \bar{\boldsymbol{H}}(x, y)\,\mathrm{e}^{-\mathrm{i}\beta z},$
  mode profile $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$, propagation constant β. (drop $^{-}$)

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vectorial mode equations, variants. (. . .)

Waveguides: Mode equations

- Where $\epsilon \cancel{(\boldsymbol{r})}$, $\mu \cancel{(\boldsymbol{r})}$: $\sim \exp(\mathrm{i}\omega t)$ (FD)

$$\Delta\tilde{\boldsymbol{E}} + k^2\epsilon\mu\tilde{\boldsymbol{E}} = 0, \quad \Delta\tilde{\boldsymbol{H}} + k^2\epsilon\mu\tilde{\boldsymbol{H}} = 0$$

$$\partial_x^2\boldsymbol{E} + \partial_y^2\boldsymbol{E} + (k^2\epsilon\mu - \beta^2)\boldsymbol{E} = 0,$$
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scalar mode equation, valid for all components of $\boldsymbol{E}$, $\boldsymbol{H}$,
to be supplemented by suitable boundary and interface conditions.

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$\leftrightsquigarrow$ Eigenvalue problem with eigenvalue β, eigenfunction $\boldsymbol{E}$, $\boldsymbol{H}$,
"$\mathsf{M}(\beta)\ (\overrightarrow{\text{profile}}) = 0$".

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- Guided modes: discrete $\beta \in \mathbb{R}$, $\displaystyle\iint S_z \,\mathrm{d}x\mathrm{d}z < \infty$. $(\epsilon, \mu \in \mathbb{R})$

Waveguides: Mode equations

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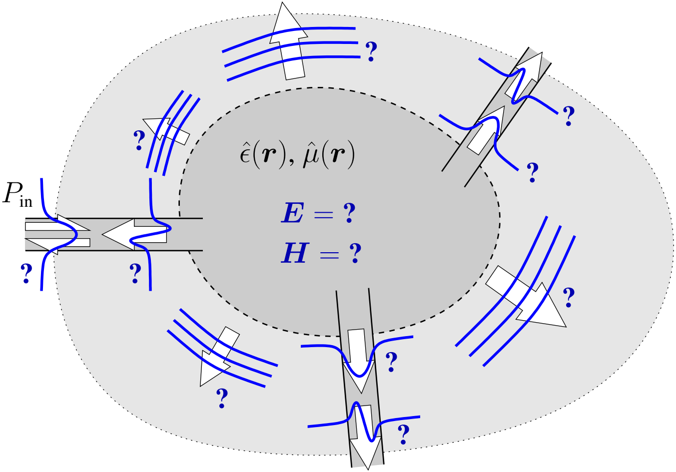
(Radiation modes: continuum of $\beta^2 \in \mathbb{R}$, oscillating external fields.)
(Leaky modes: discrete $\beta \in \mathbb{C}$, outgoing wave boundary conditions.)
(. . .)

Guided wave scattering problems

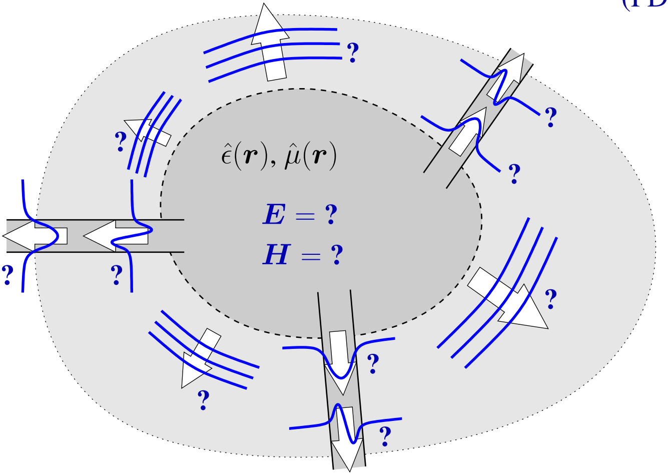
(FD)



Given external excitation $\sim \exp(\mathrm{i}\omega t),\ \omega \in \mathbb{R}$.

Resonance problems

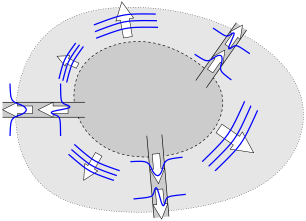
(FD . . .)



Omit excitation, look for nonzero solutions that decay in time.

Resonance problems

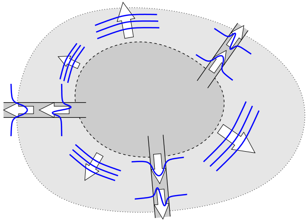
(FD ...)



$\boldsymbol{E}(\boldsymbol{r}),\ \boldsymbol{H}(\boldsymbol{r}),\ \sim \exp(\mathrm{i}\omega t),\ \omega = ?$
$\boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H},$
$\boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E},$
& outgoing wave boundary conditions.

Resonance problems

(FD . . .)



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$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H},$$
$$\boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E},$$

& outgoing wave boundary conditions.

- Look for nonzero solutions with $\omega \in \mathbb{C}$ that oscillate and decay (slowly . . .) in time.
- "$\mathsf{M}(\omega)\ (\overrightarrow{\text{field}}) = 0$", eigenvalue problem.
- Solutions: discrete eigenfrequencies ω, resonant mode profiles.

Keyword: "Quasi-Normal-Modes", QNMs.

Scalar approximation

Linear, isotropic, nonmagnetic media, $\epsilon = n^2$;
a structure with "small" variations in ϵ :

A scalar approximation may be adequate,

$$\nabla \cdot (\epsilon \boldsymbol{E}) \approx \epsilon \nabla \cdot \boldsymbol{E}$$

$$\Delta\psi - \frac{1}{c^2}\epsilon\,\ddot{\psi} = 0, \qquad \text{(TD)}$$

$$\Delta\psi + k^2\epsilon\,\psi = 0, \qquad \text{(FD)}$$

satisfied by all components ψ of $\boldsymbol{E}$, $\boldsymbol{H}$.

(Applicable to basically all types of problems.)

Beam propagation method

- Starting point: $\Delta\psi + k^2\epsilon\,\psi = 0$, $\sim \exp(\mathrm{i}\omega t)$ (FD)
  "small" changes in $\epsilon = n^2$ along a propagation coordinate z.

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  "small" changes in $\epsilon = n^2$ along a propagation coordinate z.

- Ansatz: $\qquad \psi(x,y,z) = \psi_0(x,y,z)\ \mathrm{e}^{-\mathrm{i}kn_\mathrm{r}z},$
  reference effective index n_r,
  assume that ψ_0 varies "slowly" along z $\longleftrightarrow$ neglect $\partial_z^2\psi_0$.

  $$\hookrightarrow \quad -\mathrm{i}2kn_\mathrm{r}\partial_z\psi_0 + (\partial_x^2 + \partial_y^2)\psi_0 + k^2(\epsilon - n_\mathrm{r}^2)\psi_0 = 0,$$

  PDE of first order in z, solved as an initial value problem.

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- Restriction to unidirectional propagation, reflections are neglected.
- Paraxial propagation, errors for waves with effective indices $\neq n_\mathrm{r}$.

(Many variants (vectorial, wide-angle, bi-directional, ...) have been proposed.)
(Other ways of motivating the approximation exist.)
(Term "BPM" in use also for other types of methods.)

- Keywords: Paraxial approximation,
  Slowly-varying-envelope approximation (SVEA),
  Beam propagation method (BPM).

Upcoming

Next lectures:

- Normal modes of dielectric optical waveguides, mode interference.
- Examples for dielectric optical waveguides.
- Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.