Optical Waveguide Theory (C)

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Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- Oblique semi-guided waves: 2-D integrated optics.
- Summary, concluding remarks.

. . . ?

“This concerns time harmonic fields . . . with angular frequency . . . , for vacuum wavenumber . . . , speed of light . . . , and wavelength . . . .”

“The problem is governed by the Maxwell curl equations in the frequency domain for the electric field . . . and magnetic field . . . , for (lossless) uncharged dielectric, nonmagnetic linear (isotropic) media with (piecewise constant) relative permittivity . . . :

. . . (.) ”

[M. Hammer, A. Hildebrandt, J. Förstner, *Journal of Lightwave Technology* **34**(3), 997 (2016)]

Maxwell equations, Fourier transform

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\mathrm{f}},\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\dot{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\mathrm{f}}+\dot{\boldsymbol{D}}$$

& $$\boldsymbol{F}(\boldsymbol{r},t)=\frac{1}{\sqrt{2\pi}}\int\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)\ \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega,\quad \tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)=\frac{1}{\sqrt{2\pi}}\int\boldsymbol{F}(\boldsymbol{r},t)\ \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$

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$$\nabla \cdot \boldsymbol{D} = \rho_\mathrm{f}, \quad \nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dot{\boldsymbol{D}}$$

$$\& \quad \boldsymbol{F}(\boldsymbol{r}, t) = \frac{1}{\sqrt{2\pi}} \int \tilde{\boldsymbol{F}}(\boldsymbol{r}, \omega)\ \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega, \quad \tilde{\boldsymbol{F}}(\boldsymbol{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int \boldsymbol{F}(\boldsymbol{r}, t)\ \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t$$

$$\boldsymbol{E}(\boldsymbol{r}, t),\ \boldsymbol{D}(\boldsymbol{r}, t),\ \boldsymbol{B}(\boldsymbol{r}, t),\ \boldsymbol{H}(\boldsymbol{r}, t),\ \rho_\mathrm{f}(\boldsymbol{r}, t),\ \boldsymbol{J}_\mathrm{f}(\boldsymbol{r}, t)$$

$$\leftrightsquigarrow \quad \tilde{\boldsymbol{E}}(\boldsymbol{r}, \omega),\ \tilde{\boldsymbol{D}}(\boldsymbol{r}, \omega),\ \tilde{\boldsymbol{B}}(\boldsymbol{r}, \omega),\ \tilde{\boldsymbol{H}}(\boldsymbol{r}, \omega),\ \tilde{\rho}_\mathrm{f}(\boldsymbol{r}, \omega),\ \tilde{\boldsymbol{J}}_\mathrm{f}(\boldsymbol{r}, \omega),$$

$$\nabla \cdot \tilde{\boldsymbol{D}} = \tilde{\rho}_\mathrm{f}, \quad \nabla \times \tilde{\boldsymbol{E}} = -\mathrm{i}\omega\tilde{\boldsymbol{B}}, \quad \nabla \cdot \tilde{\boldsymbol{B}} = 0, \quad \nabla \times \tilde{\boldsymbol{H}} = \tilde{\boldsymbol{J}}_\mathrm{f} + \mathrm{i}\omega\tilde{\boldsymbol{D}}$$

(Caution: arbitrary choice of $\sim \exp(\pm\mathrm{i}\,\omega t)$!).

Maxwell equations, frequency domain

$$\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{D}} = \tilde{\rho}_{\mathrm{f}}, \quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{E}} = -\mathrm{i}\omega\tilde{\boldsymbol{B}}, \quad \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}} = 0, \quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{H}} = \tilde{\boldsymbol{J}}_{\mathrm{f}} + \mathrm{i}\omega\tilde{\boldsymbol{D}}.$$

$$\boldsymbol{F}(\boldsymbol{r},t)\in\mathbb{R} \rightsquigarrow \tilde{\boldsymbol{F}}(\boldsymbol{r},-\omega) = (\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega))^*$$

Maxwell equations, frequency domain

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$$\boldsymbol{F}(\boldsymbol{r},t)\in\mathbb{R} \rightsquigarrow \tilde{\boldsymbol{F}}(\boldsymbol{r},-\omega)=(\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega))^*$$

"at frequency ω_0": $\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)=\sqrt{\frac{\pi}{2}}\,\bar{\boldsymbol{F}}(\boldsymbol{r})\,\delta(\omega-\omega_0)+\sqrt{\frac{\pi}{2}}\,\bar{\boldsymbol{F}}^*(\boldsymbol{r})\,\delta(\omega+\omega_0)$

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$$\boldsymbol{F}(\boldsymbol{r},t) = \mathrm{Re}\left\{\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t}\right\},$$

$$\text{“}\boldsymbol{F}(\boldsymbol{r},t) = \frac{1}{2}\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t} + \text{c.c.”}.$$

Maxwell equations, frequency domain

$$\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{D}} = \tilde{\rho}_{\rm f}, \quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{E}} = -\mathrm{i}\omega\tilde{\boldsymbol{B}}, \quad \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}} = 0, \quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{H}} = \tilde{\boldsymbol{J}}_{\rm f} + \mathrm{i}\omega\tilde{\boldsymbol{D}}.$$

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$$\hookrightarrow \bar{\boldsymbol{E}}(\boldsymbol{r}),\ \bar{\boldsymbol{D}}(\boldsymbol{r}),\ \bar{\boldsymbol{B}}(\boldsymbol{r}),\ \bar{\boldsymbol{H}}(\boldsymbol{r}),\ \bar{\rho}_{\rm f}(\boldsymbol{r}),\ \bar{\boldsymbol{J}}_{\rm f}(\boldsymbol{r}),\ \sim \exp(\mathrm{i}\omega_0 t),$$

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{D}} = \bar{\rho}_{\rm f}, \quad \boldsymbol{\nabla}\times\bar{\boldsymbol{E}} = -\mathrm{i}\omega_0\bar{\boldsymbol{B}}, \quad \boldsymbol{\nabla}\cdot\bar{\boldsymbol{B}} = 0, \quad \boldsymbol{\nabla}\times\bar{\boldsymbol{H}} = \bar{\boldsymbol{J}}_{\rm f} + \mathrm{i}\omega_0\bar{\boldsymbol{D}}.$$

Caution: Decorations $\tilde{}$, $\bar{}$, $_0$ are ususally omitted; context determines interpretation of symbols.

Polarization

$\tilde{\boldsymbol{P}}$: density of electric dipole moment (bound charges).

$$\tilde{\boldsymbol{D}} = \epsilon_0 \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{P}}, \qquad [\tilde{\boldsymbol{D}}] = [\tilde{\boldsymbol{P}}] = \frac{\mathrm{As\,m}}{\mathrm{m}^3}, \ [\tilde{\boldsymbol{E}}] = \frac{\mathrm{V}}{\mathrm{m}},$$

- vacuum permittivity $\epsilon_0 = 8.854187817\ldots \cdot 10^{-12} \left[\frac{\mathrm{F}}{\mathrm{m}} = \frac{\mathrm{As}}{\mathrm{Vm}}\right].$

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- Linear dielectrica:

$$\tilde{\boldsymbol{P}} = \epsilon_0 \hat{\chi}_\mathrm{e} \tilde{\boldsymbol{E}}, \qquad \hat{\chi}_\mathrm{e}\text{: dielectric susceptibility, } [\hat{\chi}_\mathrm{e}] = \hat{1}.$$

$$\hookrightarrow \tilde{\boldsymbol{D}} = \epsilon_0 (\hat{1} + \hat{\chi}_\mathrm{e}) \tilde{\boldsymbol{E}} = \epsilon_0 \hat{\epsilon} \tilde{\boldsymbol{E}}, \qquad \hat{\epsilon}\text{: relative permittivity, } [\hat{\epsilon}] = \hat{1}.$$

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- $\hat{\chi}_\mathrm{e}(\boldsymbol{r}, \omega)$, $\hat{\epsilon}(\boldsymbol{r}, \omega)$ are determined in the frequency domain.
- Complications: $\mathrm{Im}\,\epsilon$, $\hat{\epsilon}(T)$, $\hat{\epsilon}(\boldsymbol{F})$, $\chi^{(2)}_{jkl} E_k E_l$, $\chi^{(3)}_{jklm} E_k E_l E_m$, ...
- Simpler cases: $\hat{\epsilon}(\not{\boldsymbol{r}})$, $\hat{\epsilon} = \epsilon \hat{1}$.

Magnetization

$\tilde{\boldsymbol{M}}$: density of magnetic dipole moments (bound currents).

$$\tilde{\boldsymbol{H}} = \frac{1}{\mu_0}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{M}}, \qquad [\tilde{\boldsymbol{H}}] = [\boldsymbol{M}] = \frac{\mathrm{A\,m^2}}{\mathrm{m^3}}, \ [\tilde{\boldsymbol{B}}] = \mathrm{T} = \frac{\mathrm{Vs}}{\mathrm{m^2}},$$

vacuum permeability $\mu_0 = 4\pi \cdot 10^{-7} \left[\frac{\mathrm{N}}{\mathrm{A^2}} = \frac{\mathrm{Vs}}{\mathrm{Am}}\right]$.

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- Linear magnetic media:

$$\tilde{\boldsymbol{M}} = \hat{\chi}_{\mathrm{m}}\tilde{\boldsymbol{H}}, \qquad \hat{\chi}_{\mathrm{m}}:\ \text{magnetic susceptibility},\ [\hat{\chi}_{\mathrm{m}}] = \hat{1}.$$

$$\hookrightarrow \tilde{\boldsymbol{B}} = \mu_0(\hat{1} + \hat{\chi}_{\mathrm{m}})\tilde{\boldsymbol{H}} = \mu_0\hat{\mu}\tilde{\boldsymbol{H}},\ \hat{\mu}:\ \text{relative permeability},\ [\hat{\mu}] = \hat{1}.$$

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- $\hat{\chi}_{\mathrm{m}}(\boldsymbol{r}, \omega)$, $\hat{\mu}(\boldsymbol{r}, \omega)$ are determined in the frequency domain.
- Complications: manifold.
- Traditional integrated optics (frequencies, media): $\hat{\mu}(\not{\boldsymbol{r}}) = \hat{1}$.

(Material) dispersion: $\hat{\epsilon}(\boldsymbol{r},\omega)$, $\hat{\mu}(\boldsymbol{r},\omega)$ are frequency dependent.

$$\tilde{\boldsymbol{D}}(\boldsymbol{r},\omega) = \epsilon_0\hat{\epsilon}(\boldsymbol{r},\omega)\tilde{\boldsymbol{E}}(\boldsymbol{r},\omega), \quad \tilde{\boldsymbol{B}}(\boldsymbol{r},\omega) = \mu_0\hat{\mu}(\boldsymbol{r},\omega)\tilde{\boldsymbol{H}}(\boldsymbol{r},\omega)$$

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$$\hookrightarrow \quad \boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0 \int \hat{\epsilon}_{\mathrm{TD}}(\boldsymbol{r},t-t')\boldsymbol{E}(\boldsymbol{r},t')\,\mathrm{d}t',$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \int \hat{\mu}_{\mathrm{TD}}(\boldsymbol{r},t-t')\boldsymbol{H}(\boldsymbol{r},t')\,\mathrm{d}t'.$$

Helmholtz equations

Linear dielectric media without free charges or currents,
time dependence $\sim \exp(\mathrm{i}\omega t)$, fields $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{D}(\boldsymbol{r})$, $\boldsymbol{B}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$,
material properties $\hat{\epsilon}(\boldsymbol{r})$, $\hat{\mu}(\boldsymbol{r})$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega \boldsymbol{B}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega \boldsymbol{D},$$
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$$\hookrightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H}, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E}, \qquad \boldsymbol{\nabla} \cdot \hat{\epsilon}\boldsymbol{E} = 0, \quad \boldsymbol{\nabla} \cdot \hat{\mu}\boldsymbol{H} = 0.$$

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$$\hookrightarrow \quad \boldsymbol{\nabla} \times (\hat{\mu}^{-1} \boldsymbol{\nabla} \times \boldsymbol{E}) = \omega^2 \epsilon_0 \mu_0 \hat{\epsilon} \boldsymbol{E} \quad \text{or} \quad \boldsymbol{\nabla} \times (\hat{\epsilon}^{-1} \boldsymbol{\nabla} \times \boldsymbol{H}) = \omega^2 \epsilon_0 \mu_0 \hat{\mu} \boldsymbol{H}.$$

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Linear dielectric media without free charges or currents,
time dependence $\sim \exp(\mathrm{i}\omega t)$, fields $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{D}(\boldsymbol{r})$, $\boldsymbol{B}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$,
material properties $\hat{\epsilon}(\boldsymbol{r})$, $\hat{\mu}(\boldsymbol{r})$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\mathrm{i}\omega\boldsymbol{B},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\mathrm{i}\omega\boldsymbol{D},$$
$$\boldsymbol{D}=\epsilon_0\hat{\epsilon}\boldsymbol{E},\quad \boldsymbol{B}=\mu_0\hat{\mu}\boldsymbol{H}.$$

↪ $$\boldsymbol{\nabla}\times\boldsymbol{E}=-\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H},\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E},\qquad \boldsymbol{\nabla}\cdot\hat{\epsilon}\boldsymbol{E}=0,\quad \boldsymbol{\nabla}\cdot\hat{\mu}\boldsymbol{H}=0.$$

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Where $\hat{\epsilon}=\epsilon\hat{1}$, $\boldsymbol{\nabla}\epsilon=0$, $\hat{\mu}=\mu\hat{1}$, $\boldsymbol{\nabla}\mu=0$: (!)

↪ $$\Delta\boldsymbol{E}+\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\boldsymbol{E}=0\quad\text{or}\quad \Delta\boldsymbol{H}+\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\boldsymbol{H}=0,\qquad \mathrm{c}=\frac{1}{\sqrt{\epsilon_0\mu_0}}.$$

Plane harmonic waves

Where $\hat{\epsilon} = \epsilon \hat{1}$, $\nabla \epsilon = 0$, $\hat{\mu} = \mu \hat{1}$, $\nabla \mu = 0$: (!)

Components of $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy

$$\Delta \psi + \frac{\omega^2}{\mathrm{c}^2} \epsilon \mu \, \psi = 0.$$

$$\psi(\boldsymbol{r}, t) = \psi_0 \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_\mathrm{m} \cdot \boldsymbol{r} - \omega t)}, \qquad -\boldsymbol{k}_\mathrm{m}^2 + \frac{\omega^2}{\mathrm{c}^2} \epsilon \mu = 0.$$

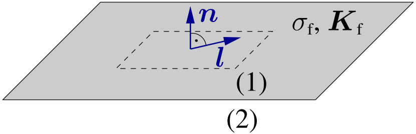
(Mixture of TD and FD expressions; $\tilde{}$, $\bar{}$, Re, $1/2$, c.c. omitted; sloppy, but common.)

- Medium: refractive index: $n = \sqrt{\epsilon \mu}$
- Periodicity in time:
  - angular frequency: ω,
  - frequency: $f = \omega/(2\pi)$,
  - period: $T = 1/f = 2\pi/\omega$,
- Spatial periodicity:
  - wave vector: $\boldsymbol{k}_\mathrm{m}$, $k_\mathrm{m} = |\boldsymbol{k}_\mathrm{m}|$,
  - wavenumber: $k_\mathrm{m} = \omega/\mathrm{c}_\mathrm{m} = (\omega/\mathrm{c})n = k\,n$,
  - vacuum wavenumber: $k = \omega/\mathrm{c}$,
  - vacuum wavelength: $\lambda = 2\pi/k = 2\pi\mathrm{c}/\omega$,
  - wavelength in the medium: $\lambda_\mathrm{m} = 2\pi/k_\mathrm{m} = 2\pi/(kn) = \lambda/n$.
- Phase velocity:
  - speed of light in vacuum: $\mathrm{c} = 1/\sqrt{\epsilon_0 \mu_0} = \lambda f$,
  - in the medium: $\mathrm{c}_\mathrm{m} = \mathrm{c}/n = \lambda_\mathrm{m} f$.

(Use of symbols depends highly on context.)

Electromagnetic spectrum ▶ ▶

Interface conditions



Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, surface charge density σ_f, surface current density $\boldsymbol{K}_\mathrm{f}$:

$$\boldsymbol{n} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \sigma_\mathrm{f}, \qquad \boldsymbol{l} \cdot (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0,$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0, \qquad \boldsymbol{l} \cdot (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{l} \cdot (\boldsymbol{K}_\mathrm{f} \times \boldsymbol{n}).$$

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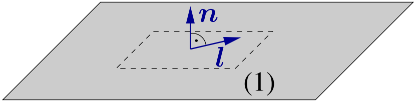

Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, surface without free charges or currents:

$$\boldsymbol{n} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = 0, \qquad \boldsymbol{l} \cdot (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0,$$
$$\boldsymbol{n} \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0, \qquad \boldsymbol{l} \cdot (\boldsymbol{H}_1 - \boldsymbol{H}_2) = 0\,.$$

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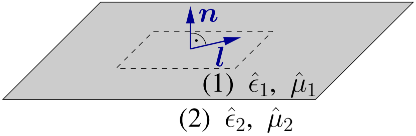


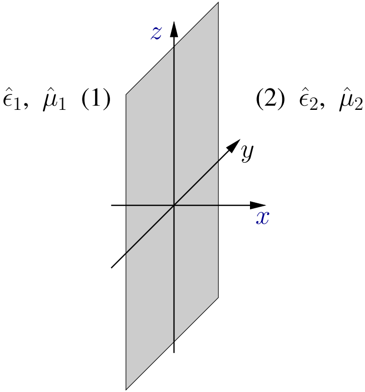
Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, linear media with permittivities $\hat{\epsilon}_1$, $\hat{\epsilon}_2$, and permeabilities $\hat{\mu}_1$, $\hat{\mu}_2$:

$$\boldsymbol{n} \cdot (\hat{\epsilon}_1 \boldsymbol{E}_1 - \hat{\epsilon}_2 \boldsymbol{E}_2) = 0, \qquad \boldsymbol{l} \cdot (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0,$$

$$\boldsymbol{n} \cdot (\hat{\mu}_1 \boldsymbol{H}_1 - \hat{\mu}_2 \boldsymbol{H}_2) = 0, \qquad \boldsymbol{l} \cdot (\boldsymbol{H}_1 - \boldsymbol{H}_2) = 0.$$

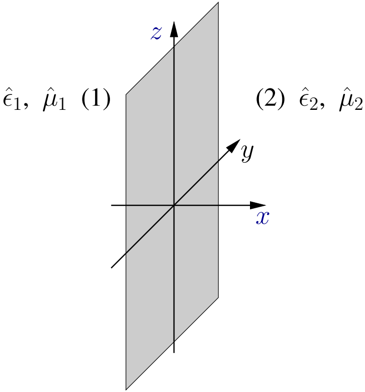
Reflection and transmission of plane waves at dielectric interfaces

(FD)



- $\boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat\mu\,\boldsymbol{H}$,
  $\boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat\epsilon\,\boldsymbol{E}$
- $\hat\epsilon(\boldsymbol{r})$ and $\hat\mu(\boldsymbol{r})$ are constant along y, z

(FD)



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↪ $$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}'(x)\,\mathrm{e}^{-\mathrm{i}(k_y y + k_z z)},\quad \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{H}'(x)\,\mathrm{e}^{-\mathrm{i}(k_y y + k_z z)}$$

1-D problem for $\boldsymbol{E}'$, $\boldsymbol{H}'$.

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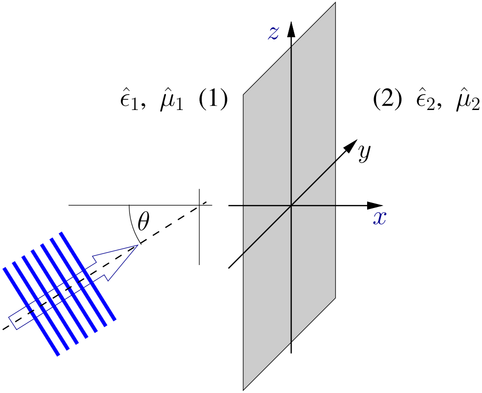

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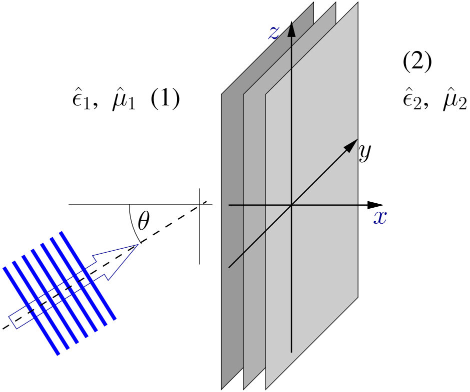
1-D problem for $\boldsymbol{E}'$, $\boldsymbol{H}'$.

(incoming plane wave at angle θ)
(orient coordinates ($k_y = 0$), plane of incidence, distinguish polarizations)
(write ansatz functions for incoming, reflected, and transmitted waves)
(interface conditions determine the amplitudes)

↪ Fresnel equations.

Dielectric multilayer structures

(FD)



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1-D problem for $\boldsymbol{E}'$, $\boldsymbol{H}'$.

(. . . .)
(. . . .)
(. . . .)
(. . . .)

↪ Reflectance and transmittance properties. ▶

Energy of electromagnetic fields

(TD)

- Force on a particle with charge q, velocity $\boldsymbol{v}$, in a field $\boldsymbol{E}, \boldsymbol{B}$:
  $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$,

Energy of electromagnetic fields

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- work for shifting the particle by $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$:
  $\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}\,\mathrm{d}t = q\boldsymbol{E} \cdot \boldsymbol{v}\,\mathrm{d}t,$

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For a charge density $\rho_\mathrm{f}(\boldsymbol{r}, t)$:

force density $\boldsymbol{f} = \rho_\mathrm{f}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$,

power density $\boldsymbol{f} \cdot \boldsymbol{v} = \rho_\mathrm{f}\boldsymbol{E} \cdot \boldsymbol{v} = \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}$,

total work per time unit done in $\mathcal{V}$: $\dfrac{\mathrm{d}W_\mathcal{V}}{\mathrm{d}t} = \int_\mathcal{V} \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}\,\mathrm{d}\mathcal{V}$.

Power & energy density, Poynting theorem

(TD)

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dot{\boldsymbol{D}}, \quad \nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} W_{\mathcal{V}}^{\mathrm{mech}} = \int_{\mathcal{V}} \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E} \, \mathrm{d}\mathcal{V} = - \int_{\mathcal{V}} (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}}) \, \mathrm{d}\mathcal{V} - \int_{\mathcal{V}} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) \, \mathrm{d}\mathcal{V},$$

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  $\rightsquigarrow \quad \dot{w} = (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}})$

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  $\rightsquigarrow \dot{w} = (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}})$

$\mathcal{V}$ arbitrary

$$\hookrightarrow \dot{w} + \nabla \cdot \boldsymbol{S} = -\boldsymbol{J}_{\mathrm{f}} \cdot \boldsymbol{E}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(W_{\mathcal{V}}^{\mathrm{mech}} + W_{\mathcal{V}}^{\mathrm{field}}\right) = -\oint_{\partial\mathcal{V}} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a}.$$

Electromagnetic energy, frequency domain

Lossless uncharged nondispersive (...) linear media:

$$w = \frac{1}{2}(\epsilon_0 \boldsymbol{E} \cdot \hat{\epsilon}\boldsymbol{E} + \mu_0 \boldsymbol{H} \cdot \hat{\mu}\boldsymbol{H}), \quad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}, \quad \dot{w} + \boldsymbol{\nabla} \cdot \boldsymbol{S} = 0,$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\, \tilde{\boldsymbol{E}}(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\omega t}, \quad \boldsymbol{H}(\boldsymbol{r}, t) = \mathrm{Re}\, \tilde{\boldsymbol{H}}(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\omega t}$$

↪ $\boldsymbol{S}$, w oscillate in time.

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↪ $\boldsymbol{S}$, w oscillate in time.

(FD)

Consider time-averaged quantities: $\bar{f}(t) = \frac{1}{T} \int_t^{t+T} f(t')\, \mathrm{d}t'$

(exercise)

↪ $\overline{w} = \frac{1}{4} \mathrm{Re}\, \left(\epsilon_0 \tilde{\boldsymbol{E}}^* \cdot \hat{\epsilon}\, \tilde{\boldsymbol{E}} + \mu_0 \tilde{\boldsymbol{H}}^* \cdot \hat{\mu}\, \tilde{\boldsymbol{H}}\right), \quad \overline{\boldsymbol{S}} = \frac{1}{2} \mathrm{Re}\, \left(\tilde{\boldsymbol{E}}^* \times \tilde{\boldsymbol{H}}\right).$

$\dot{\overline{w}} = \overline{\dot{w}} = 0, \quad \overline{\boldsymbol{\nabla} \cdot \boldsymbol{S}} = \boldsymbol{\nabla} \cdot \overline{\boldsymbol{S}} \quad \rightsquigarrow \quad \boldsymbol{\nabla} \cdot \overline{\boldsymbol{S}} = 0, \quad \oint_{\mathcal{V}} \overline{\boldsymbol{S}} \cdot \mathrm{d}\boldsymbol{a} = 0;$

"power balance", conservation of energy.

Wave propagation in attenuating media

Specifically: homogeneous isotropic conductors, linear media.

Electric field drives the free currents:

Ohm's law $\boldsymbol{J}_{\mathrm{f}} = \sigma \boldsymbol{E}$, σ: conductivity of the material.

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$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_\mathrm{f}, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \sigma \boldsymbol{E} + \dot{\boldsymbol{D}}.$$

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$$\hookrightarrow \quad \dot{\rho}_\mathrm{f} = -\frac{\sigma}{\epsilon_0 \epsilon} \rho_\mathrm{f}, \quad \rho_\mathrm{f}(\boldsymbol{r}, t) = \rho_\mathrm{f}(\boldsymbol{r}, t_0) \exp\left(-\frac{\sigma}{\epsilon_0 \epsilon}(t - t_0) \right),$$

assume $\rho_\mathrm{f}(\boldsymbol{r}, t_0) = 0 \rightsquigarrow \rho_\mathrm{f}(\boldsymbol{r}, t) = 0 \quad \forall t.$

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$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\mu_0 \mu \dot{\boldsymbol{H}}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{H} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \sigma \boldsymbol{E} + \epsilon_0 \epsilon \dot{\boldsymbol{E}}.$$

Telegrapher equation

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \times \boldsymbol{E} = -\mu_0 \mu \dot{\boldsymbol{H}}, \quad \nabla \cdot \boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \epsilon_0 \epsilon \dot{\boldsymbol{E}}$$

$$\hookrightarrow \Delta \boldsymbol{E} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\boldsymbol{E}} - \mu_0 \mu \sigma \dot{\boldsymbol{E}} = 0, \quad \Delta \boldsymbol{H} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\boldsymbol{H}} - \mu_0 \mu \sigma \dot{\boldsymbol{H}} = 0,$$

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Telegrapher equation.

Frequency domain: $\boldsymbol{E}(\boldsymbol{r}, t) = \tilde{\boldsymbol{E}}(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\omega t}$,

$$\hookrightarrow \Delta \tilde{\boldsymbol{E}} + \left(\frac{\omega^2}{\mathrm{c}^2} \epsilon \mu - \mathrm{i} \omega \mu_0 \mu \sigma \right) \tilde{\boldsymbol{E}} = 0.$$

Nonconducting media $\sigma = 0, \quad \Delta \tilde{\boldsymbol{E}} + \left(\frac{\omega^2}{\mathrm{c}^2} \epsilon \mu \right) \tilde{\boldsymbol{E}} = 0.$

Telegrapher equation

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \times \boldsymbol{E} = -\mu_0 \mu \dot{\boldsymbol{H}}, \quad \nabla \cdot \boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \epsilon_0 \epsilon \dot{\boldsymbol{E}}$$

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Nonconducting media $\sigma = 0, \quad \Delta \tilde{\boldsymbol{E}} + \left(\frac{\omega^2}{\mathrm{c}^2} \epsilon \mu \right) \tilde{\boldsymbol{E}} = 0.$

Define $\bar{\epsilon}$ such that $\frac{\omega^2}{\mathrm{c}^2} \bar{\epsilon} \mu = \frac{\omega^2}{\mathrm{c}^2} \epsilon \mu - \mathrm{i} \omega \mu_0 \mu \sigma$, i.e. $\bar{\epsilon} = \epsilon - \mathrm{i} \frac{\sigma}{\epsilon_0 \omega}$

$$\hookrightarrow \Delta \tilde{\boldsymbol{E}} + k^2 \bar{\epsilon} \mu \tilde{\boldsymbol{E}} = 0,$$ Helmholtz equation, $\bar{\epsilon} \in \mathbb{C}$, $\quad k = \frac{\omega}{\mathrm{c}}$.

Wave attenuation

$$\Delta\tilde{\boldsymbol{E}} + k^2\bar{\epsilon}\mu\tilde{\boldsymbol{E}} = 0, \quad \bar{\epsilon} \in \mathbb{C}$$

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↪ solutions $\sim \mathrm{e}^{-\mathrm{i}(k\bar{n}z - \omega t)}$

with refractive index $\bar{n} = n' - \mathrm{i}n'' = \sqrt{\bar{\epsilon}\mu} \in \mathbb{C},$ (!)

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$$\mathrm{e}^{-\mathrm{i}(k\bar{n}z - \omega t)} = \mathrm{e}^{-\mathrm{i}(kn'z - \omega t)}\,\mathrm{e}^{-kn''z},$$

damped plane wave solutions

↭ sign of n'', choice of $\exp(\pm\mathrm{i}\omega)$.

Issues:

- penetration depth,
- $\boldsymbol{S}$ and w decay with z,
- still transverse waves,
- $\boldsymbol{E}$, $\boldsymbol{H}$ no longer in phase,
- notions of wavenumber, wavelength, phase velocity $\in \mathbb{C}$.

A typical setting:

- "uncharged dielectric medium": $\not{q_\mathrm{f}}$, $\not{\boldsymbol{J}_\mathrm{f}}$.
- "linear medium": $\boldsymbol{D} = \epsilon_0 \hat{\epsilon} \boldsymbol{E}$, $\boldsymbol{B} = \mu_0 \hat{\mu} \boldsymbol{H}$.
- "isotropic medium": $\hat{\epsilon} = \epsilon\, \hat{1}$, $\hat{\mu} = \mu\, \hat{1}$.
- "nonmagnetic medium": $\hat{\mu} = \hat{1}$.
- "lossless medium": $\hat{\epsilon}^\dagger = \hat{\epsilon}$, $\hat{\mu}^\dagger = \hat{\mu}$, $(\epsilon, \mu \in \mathbb{R})$.
- "piecewise constant" $\rightarrow$ "dependent on position".
- "electric and magnetic field": eliminate $\boldsymbol{D}$ and $\boldsymbol{B}$, retain $\boldsymbol{E}$ and $\boldsymbol{H}$.
- "governed by the curl equations": divergence eqns. are satisfied.
- "frequency domain, time harmonic fields, frequency, wavelength": ... as discussed.

Upcoming

Next lectures:

- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.
- Examples for dielectric optical waveguides.