1. *Parametrized ringresonator model*

   Fill in some details of the scattering-matrix model of a ringresonator device as discussed in lecture H. Given the scattering matrices for the couplers, expressions for the field evolution along the cavity bends, and the external input amplitudes,

   (a) show that the model predicts output amplitudes as given by the expressions on sheet H.8, and

   (b) show that the model leads to the expressions for the transmitted and dropped power on sheet H.9.

2. *Quality factor of an open optical cavity*

   According to the textbook [J.D. Jackson, *Classical Electrodynamics*, 3rd. ed., Wiley, New York (1998)], the quality factor of an open cavity is defined as

$$Q = 2\pi \, \frac{\text{time averaged energy stored in the cavity}}{\text{energy loss per cycle}} \,. \tag{1}$$

   Denote the time averaged energy stored in the cavity by $U(t)$. Then, per (time) cycle, i.e. per time period $T = 2\pi/\omega$, for angular frequency ω, the energy loss of the cavity is $-\dfrac{\mathrm{d}U}{\mathrm{d}t}T$.

   (a) Show that the energy stored in the cavity depends on time as

$$U(t) = U_0 \, \mathrm{e}^{-\dfrac{\omega}{Q}t} = U_0 \, \mathrm{e}^{-\dfrac{t}{\tau}}, \tag{2}$$

   where $\tau = Q/\omega$ can be identified as the lifetime for photons trapped in the cavity.

   (b) Assuming that the electric field (spatial dependence and vector character suppressed) in the cavity is of the form

$$E(t) = E_0 \, \mathrm{e}^{\mathrm{i}\omega_\mathrm{c} t}, \tag{3}$$

   with complex eigenfrequency $\omega_\mathrm{c} = \omega + \mathrm{i}\alpha$, and assuming that the squared electric field is proportional to the energy density $U \sim |E|^2$, show that the Q-factor and the photon lifetime are related to the real and imaginary parts of the complex eigenfrequency as

$$Q = \frac{\omega}{2\alpha}, \quad \text{or} \quad \tau = \frac{1}{2\alpha}. \tag{4}$$

   (c) Assume that the electric field in the cavity oscillates in time according to Eq. (3) for $t > 0$, with $E = 0$ for $t < 0$. Show that the power spectrum of the field in the cavity, and correspondingly the power spectrum of the outgoing radiation, is

$$P(\omega') \sim |\tilde{E}(\omega')|^2 \sim \frac{1}{\alpha^2 + (\omega' - \omega)^2} \,. \tag{5}$$

   (d) Show that the spectral Full-Width-at-Half-Maximum (FWHM) $\Delta\omega$ of the outgoing radiation is

$$\Delta\omega = 2\alpha = \frac{\omega}{Q} = \frac{1}{\tau}. \tag{6}$$

   Hence we have related the above definition of the Q-factor to the relation $Q = \omega/(2\alpha)$ as introduced for the whispering gallery modes in lecture H (H.26; note the different use of symbols!).

   (e) According to Eq. (6), the Q-factor can be written as the ratio of the frequency and the spectral width of the outgoing radiation:

$$Q = \frac{\omega}{\Delta\omega}. \tag{7}$$

   Translate that expression to wavelengths, i.e. show that $Q = \lambda/\Delta\lambda$, where λ is the resonance wavelength, and $\Delta\lambda$ is the FWHM (in wavelengths) of the outgoing radiation.

3. *Coupled mode theory*

Refer to the setting for coupled mode theory as discussed in lecture I. Fill in some details of the derivations, which were omitted in the lecture.

(a) Derive the general coupled mode equations as stated below the lines on sheets I.17 and I.18. Familiarize yourself with the setting and the ansatz for the coupled mode field (I.16), then choose either the reciprocity identity (I.17) or the functional (I.18) as a starting point for your derivation. You might wish to follow the hints given in small print on the sheets.

(b) Show that the assumptions on longitudinal homogeneity of the basis structures and of the full structure leads to the simpler system of coupled mode equations (constant coefficients) as stated on sheet I.20.

(c) Verify that the expression given on the second half of sheet I.23 (first full overlay) is indeed a solution of the coupled ordinary differential equations for the system with two orthogonal coupled modes.

Hand in your solutions until Wednesday, July 10, 09:15. Good luck!