

1. Guided 2-D TE/TM modes, orthogonality properties

Reconsider the orthogonality properties of guided modes, as discussed in the lecture, now for a set of non-degenerate TE- and TM-polarized modes supported by a 2-D waveguide configuration with permittivity $\epsilon(x)$. These are electromagnetic fields of the form

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x) e^{-i\beta_m z} \tag{1}$$

with mode profiles $\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m$ and pairwise different propagation constants β_m . We introduce an abbreviation $\psi_m^p = (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m)$ for the mode profile; superscripts $p = \text{TE}, \text{TM}$ denote the polarization. Similar to the 3-D case, a product

$$(\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2) := \frac{1}{4} \int (E_{1x}^* H_{2y} - E_{1y}^* H_{2x} + H_{1y}^* E_{2x} - H_{1x}^* E_{2y}) dx \tag{2}$$

of two electromagnetic fields $(\mathbf{E}_j, \mathbf{H}_j), j = 1, 2$, will be used.

- (a) Recall the specific modal properties of the TE/TM modes supported by straight 2-D waveguides.
- (b) Show that the time averaged power P_m per lateral (y) unit length carried by mode ψ_m^p, β_m can be given the form

$$P_m := \int S_z dx = (\psi_m^p; \psi_m^p) = \begin{cases} \frac{\beta_m}{2\omega\mu_0} \int |E_{m,y}|^2 dx, & \text{if } p = \text{TE}, \\ \frac{\beta_m}{2\omega\epsilon_0} \int \frac{1}{\epsilon} |H_{m,y}|^2 dx, & \text{if } p = \text{TM}. \end{cases} \tag{3}$$

- (c) Starting from the identity

$$\nabla \cdot (\mathbf{E}_l^* \times \mathbf{H}_m + \mathbf{E}_m \times \mathbf{H}_l^*) = 0 \quad \text{for all } l, m, \tag{4}$$

and taking into account the vanishing y -derivatives of all fields in the 2-D setting, conclude that

$$(\beta_l - \beta_m) \int (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m + \bar{\mathbf{E}}_m \times \bar{\mathbf{H}}_l^*)_z dx = 0. \tag{5}$$

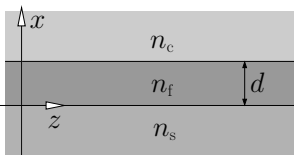
- (d) Verify that the orthogonality properties of the modal set in question can be stated in the form

$$\begin{aligned} (\psi_l^{\text{TE}}; \psi_m^{\text{TM}}) &= 0, & (\psi_l^{\text{TM}}; \psi_m^{\text{TE}}) &= 0, \\ (\psi_l^{\text{TE}}; \psi_m^{\text{TE}}) &= \frac{\beta_m}{2\omega\mu_0} \int E_{l,y}^* E_{m,y} dx = \delta_{lm} P_m, \\ (\psi_l^{\text{TM}}; \psi_m^{\text{TM}}) &= \frac{\beta_m}{2\omega\epsilon_0} \int \frac{1}{\epsilon} H_{l,y}^* H_{m,y} dx = \delta_{lm} P_m, \end{aligned} \tag{6}$$

for pairs of modes with indices l, m , with $\delta_{lm} = 1$, if $l = m$, and $\delta_{lm} = 0$, otherwise. Note that the statements still hold (?) in case of a degeneracy between modes of different polarization.

2. Three layer slab waveguide, transverse resonance condition

Specialize the model of 2-D of dielectric multilayer slab waveguides, as discussed in the lecture, to a three-layer structure as shown in the figure.



A not necessarily symmetric three layer slab waveguide, with a core thickness d , and refractive indices n_s, n_f , and n_c for the substrate, film, and cover layers, described in Cartesian coordinates x and z .

We use the notation as introduced in the lecture. Interest is in guided TE/TM modes with effective indices n_{eff} from the interval $n_s, n_c < n_{\text{eff}} < n_f$, and with propagation constants $\beta = kn_{\text{eff}}$.

(a) Verify that the profile of the principal field component ϕ can be stated in the form

$$\phi(x) = \begin{cases} B_c e^{-\kappa_c x}, & \text{for } d < x, \\ A_f \sin(\kappa_f x) + B_f \cos(\kappa_f x), & \text{for } 0 < x < d, \\ A_s e^{\kappa_s x}, & \text{for } x < 0, \end{cases} \quad (7)$$

where the local wavenumbers in x -direction are

$$\kappa_s = \sqrt{\beta^2 - k^2 n_s^2}, \quad \kappa_f = \sqrt{k^2 n_f^2 - \beta^2}, \quad \kappa_c = \sqrt{\beta^2 - k^2 n_c^2}.$$

(b) Use the conditions of continuity of ϕ and continuity of $\eta \partial_x \phi$ at $x = 0$ and at $x = d$ to establish a system of equations for the local coefficients A_s , A_f , B_f , and B_c . Here η distinguishes the polarizations as $\eta(x) = 1$ (TE) and $\eta(x) = 1/n^2(x)$ (TM); introduce quantities η_s , η_f , η_c accordingly.

(c) By eliminating successively first A_f and B_f , then B_c , and finally A_s (requiring a nonzero solution, i.e. a nonzero A_s), show that the condition for the existence of guided modes can be written as a transcendental equation

$$\tan(\kappa_f d) = \frac{(\eta_s \kappa_s + \eta_c \kappa_c) \eta_f \kappa_f}{\eta_f^2 \kappa_f^2 - \eta_s \kappa_s \eta_c \kappa_c}, \quad (8)$$

the so-called *transverse resonance condition* for the waveguide.

(d) Check whether the condition (8) is satisfied for the effective index values as given in the lecture for the example with waveguide parameters $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.0$, $d = 1.5 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$.

3. Functional for guided modes of 3-D dielectric waveguides

Consider a nonmagnetic, lossless, and isotropic 3-D dielectric waveguide with a cross section spanned by the x - and y -plane, with a real scalar permittivity $\epsilon(x, y)$ that is constant along the axis z of the waveguide. Guided modes supported by that waveguide are electromagnetic fields of the form

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z}, \quad (9)$$

with a real propagation constant β and with mode profile components $\bar{\mathbf{E}}$, $\bar{\mathbf{H}}$ that vanish for $x, y \rightarrow \pm\infty$. The Maxwell curl equations in the frequency domain, for a time dependence $\sim e^{i\omega t}$ with angular frequency ω , need to be satisfied. For a field in the modal form of Eq. (9), these can be stated as:

$$\begin{aligned} (C + i\beta R)\bar{\mathbf{E}} &= -i\omega\mu_0\bar{\mathbf{H}}, \\ (C + i\beta R)\bar{\mathbf{H}} &= i\omega\epsilon_0\epsilon\bar{\mathbf{E}}, \end{aligned} \quad \text{with } R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}. \quad (10)$$

For simplicity we restrict things to a smooth refractive index profile, such that the permittivity and all fields can be assumed to be continuous, with continuous derivatives. Now consider the functional

$$\mathcal{B}(\mathbf{E}, \mathbf{H}) = \frac{\omega\epsilon_0\langle\mathbf{E}, \epsilon\mathbf{E}\rangle + \omega\mu_0\langle\mathbf{H}, \mathbf{H}\rangle + i\langle\mathbf{E}, \mathbf{C}\mathbf{H}\rangle - i\langle\mathbf{H}, \mathbf{C}\mathbf{E}\rangle}{\langle\mathbf{E}, \mathbf{R}\mathbf{H}\rangle - \langle\mathbf{H}, \mathbf{R}\mathbf{E}\rangle}, \quad (11)$$

for an inner product $\langle\mathbf{F}, \mathbf{G}\rangle = \iint \mathbf{F}^* \cdot \mathbf{G} \, dx \, dy$.

(a) Show that the functional evaluates to the propagation constant, if a valid mode profile is inserted, i.e. show that $\mathcal{B}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta$ for fields $\bar{\mathbf{E}}$, $\bar{\mathbf{H}}$ that satisfy Eq. (10).

(b) Show that the functional becomes stationary at a valid mode profile, i.e. show that

$$\left. \frac{d}{ds} \mathcal{B}(\bar{\mathbf{E}} + s\delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + s\delta\bar{\mathbf{H}}) \right|_{s=0} = 0 \quad (12)$$

for fields $\bar{\mathbf{E}}$, $\bar{\mathbf{H}}$ that satisfy Eq. (10), and for arbitrary variations $\delta\bar{\mathbf{E}}$, $\delta\bar{\mathbf{H}}$ (with real s). This implies: If the functional becomes stationary at some field \mathbf{E} , \mathbf{H} , i.e. if Eq. (12) is satisfied for arbitrary $\delta\bar{\mathbf{E}}$, $\delta\bar{\mathbf{H}}$, then \mathbf{E} , \mathbf{H} satisfy Eq. (10).

Hints: Observe the ‘‘cc-bilinearity’’ of $\langle \cdot, \cdot \rangle$. You might wish to show first that, for fields \mathbf{F} , \mathbf{G} where this is applicable, $\langle\mathbf{F}, \mathbf{R}\mathbf{G}\rangle = -\langle\mathbf{R}\mathbf{F}, \mathbf{G}\rangle$, and $\langle\mathbf{F}, \mathbf{C}\mathbf{G}\rangle = \langle\mathbf{C}\mathbf{F}, \mathbf{G}\rangle$. Try to work, as far as possible, on the level of the inner products; introduce abbreviations, where appropriate, e.g. for the numerator and the denominator of \mathcal{B} .

Hand in your solutions until Wednesday, June 12, 09:15. Good luck!