Calculating Spontaneous Emission for Layered Structures

Contents

Chapter 1

Introduction

1.1 Background and Motivation

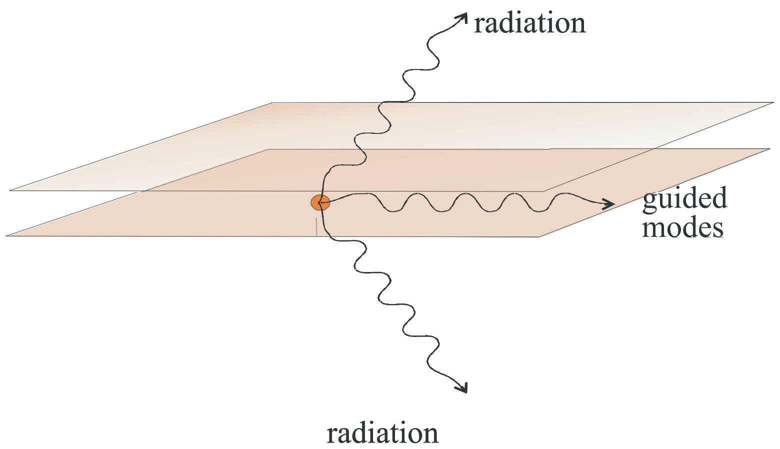

Figure 1.1: *Spontaneous emission in a layered system with a dipole source*

The local density of modes plays a dominant role in spontaneous emission of excited atoms and molecules. Over the years, various papers have been published on spontaneous emission in planar structures presenting results including radiation modes [8] only, or approximate expressions [2, 6]. To best of our knowledge no full treatment including the emission to the guided modes has been presented so far. In principle exact analytical expressions could be obtained by using Greens functions techniques, but these are normally quite

cumbersome to evaluate and usually not very transparent. Therefore, it is interesting to look for simple, complete and exact expressions.
According to the correspondence principle [8], expressions for the local density of modes can obtained by considering the energy flow into radiation and guided modes from a classical dipole positioned in one of the layers:

$$\frac{\rho}{\rho_0} = \frac{P}{P_0},$$

where ρ is the local density of modes/states of a specific structure and P the outgoing power for the same structure. ρ_0 and P_0 are the local density of modes/states and the outgoing power for a uniform structure, respectively.
The problem corresponds to a inhomogeneous Helmholtz equation with a localized source term.

In this MSc thesis the spontaneous emission of a single source embedded in a layered (planar) structure is treated (see figure (1.1)). As mostly the electric dipole moments are much stronger than the magnetic dipole moments we assume radiating electric dipoles only. The approaches for the electric dipoles and the magnetic dipoles are essentially similar.

1.2 Decay of Energy of an Excited Electronic State

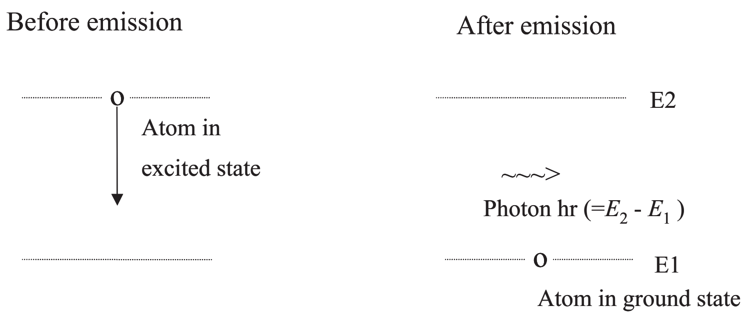

Figure 1.2: *An energy level diagram illustrating the process of spontaneous emission*

There are two means in which an atom in the excited state can decay [5]. In the first one, an atom in an excited state makes a transition to a lower state, with the emission of a photon. The photon energy is equal to the energy difference of the two atomic states. This is *spontaneous emission* or *radiative transition* (see figure (1.2)).

In second kind of decay, an atom in an excited state reaches the ground state without radiative emission, e.g. by giving off all the energy to the phonon system or by radiationless transfer to another centre. This is a *non-radiatve transition.*

The decay of an excited state can be summarized symbolically as

$$\frac{1}{\tau} = \left(\frac{1}{\tau}\right)_{radiative} + \left(\frac{1}{\tau}\right)_{nonradiative},$$

where τ is the lifetime of the atom.
In this study we are only interested in the radiative transition of light (spontaneous emission). The radiative transition probability $\left(\frac{1}{\tau}\right)_{radiative}$ is proportional to the local density of modes

$$\left(\frac{1}{\tau}\right)_{radiative} \propto \rho(\omega, \vec{r}, \text{structure}, ...),$$

where ρ is the local density of modes/states, ω is the optical frequency and $\vec{r}$ is the position of the source.

1.3 Objectives

We shall now define the objectives of this MSc thesis work in more detail

- Deriving analytical expressions for the amplitudes of the radiation and guided fields of the twodimensional problem. Both TE and TM polarization will be considered. This way we treat most of the difficulties that we might face when solving the full vectorial problem.

- Extending the analytical expressions of the twodimensional problem to the threedimensional problem.

- Deriving analytical expressions for the radiated and the guided power in the case of the threedimensional problem.
- Verifying and investigating the validity of the obtained expressions. Moreover providing some examples.

All our attention will be paid to obtain expressions for the radiated and for the guided power. That means we are not treating the evanescent waves, as they do not contribute to the outgoing power.
According to the proportionately between the transferred power P and the density of modes ρ, in this way we can directly predict the relative decay probability of atoms in a layered system.

1.4 Outline of the thesis

The structure of this MS.c thesis is as follows:
Following the introduction, in the second chapter, we will solve twodimensional inhomogeneous Helmholtz equation with a localized source term in the Fourier domain in order to find exact analytical expressions for the amplitudes of the radiation and the guided modes.
In the third chapter we will extend the expressions that have been obtained in the previous chapter to full vectorial equations by following a similar reasoning. Moreover, analytical expressions for the radiated and the guided power will be obtained.
In the fourth chapter, we investigate the validity of the obtained expressions. In the remaining part of chapter four applications to some interesting structures are given.
In the final chapter the main conclusions are presented.

Chapter 2

2D Problem

In this chapter we derive analytical expressions for the amplitudes of the radiation and the guided fields due to a radiating dipole in the core of a 2D dielectric layered structure. One of these layers may contains distributed sources. Here we consider only one dipole source as shown in figure (2.1). Both TE and TM polarizations are considered. It is assumed that the refractive index varies only along x. It should be noted that the time dependence e^{iwt} is implicit through the analysis, where ω is the frequency of the point source.

2.1 The Problem

In this section we formulate the wave equations for the layered slab media as shown in figure (2.1). Taking into account the fact that we treat a dielectric optical waveguide, we assume a permittivity and permeability of the form $\varepsilon = \varepsilon_0 n^2(x)$ and $\mu = \mu_0$, where $n(x)$ represent the spatially dependent refractive index, ε_0 and μ_0 are the free-space permittivity and free-space permeability, respectively.
From Maxwell's curl equations we obtain

$$\nabla \times \vec{E} = -i\omega\mu_0 \vec{H} \tag{2.1}$$

$$\nabla \times \vec{H} = i\omega\varepsilon_0 n^2 \vec{E} + i\omega \vec{P} \tag{2.2}$$

where $\vec{E}$ is the electric field, $\vec{H}$ is the magnetic field and $\vec{P}$ is the polarization of the source (radiating dipole).

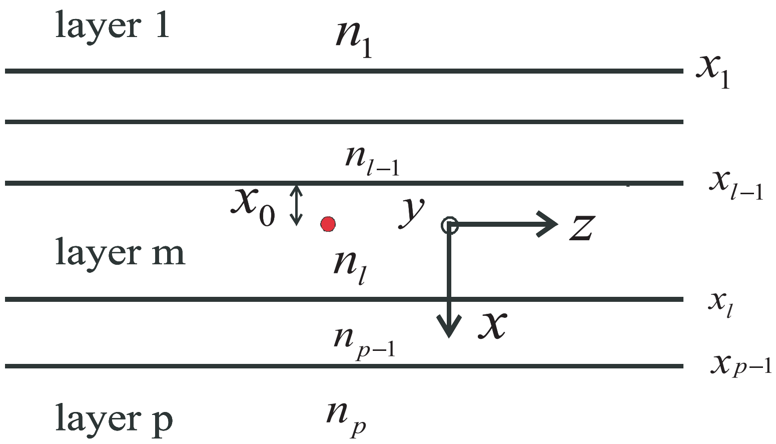

Figure 2.1: *Layered system with a dipole source.*

Now if we take the curl of Eqs (2.1) and (2.2) we find

$$\nabla \times \nabla \times \vec{E} = k_0^2 n^2(x)\vec{E} + \omega^2 \mu_0 \vec{P}, \tag{2.3}$$

$$n^2(x)\nabla \times \frac{1}{n^2(x)}\nabla \times \vec{H} = k_0^2 n^2(x)\vec{H} + i\omega n^2(x)\nabla \times \frac{\vec{P}}{n^2(x)}. \tag{2.4}$$

Since we will treat the Eq. (2.3) and (2.4) locally (for each layer) the second term in the right hand side of (2.4) can be written as

$$i\omega\nabla \times \vec{P}.$$

We consider a dipole source, therefore, the polarization can be written as

$$\vec{P} = \vec{p}\delta(\vec{r} - \vec{r}_0) \tag{2.5}$$

where $\vec{p}$ is so-called electric dipole moment.
It should be noted that the polarization is different in 2D formulation ($\partial_y = 0$) from the 3D formulation. In the following we denote the polarization regarding the 2D problem by $\vec{P}_{2D}$, where $\vec{P}_{2D} = \vec{p}\delta(x - x_0)\delta(z - z_0)$, and the one regarding the 3D problem by $\vec{P}$ (see chapter 3).

2.2 TE Polarization

For the TE polarization only the polarization along y contributes. From Eqs. (2.3) and (2.4), using $\partial_y = 0$, the considered 2D wave equation for TE polarization reads

$$[\partial_{xx} + \partial_{zz} + k_0^2 n^2(x)]E_y(x,z) = -\omega^2 \mu_0 p_{2D,y} \delta(\vec{r} - \vec{r_0}) \tag{2.6}$$

where $\vec{p}_{2D,y}$ is the electric dipole along y and $\vec{r_0}$ is the position of the source,

$$\vec{r} = \begin{pmatrix} x \\ z \end{pmatrix}.$$

We can write

$$\delta(\vec{r} - \vec{r_0}) = \delta(x - x_0)\delta(z - z_0). \tag{2.7}$$

Below we will derive expressions for the outgoing fields which may be in the form of the radiation and guided fields.

2.2.1 Radiation fields

First we determine the amplitudes of the radiation fields. Physically we expect waves to propagate a way from the source generating them and not towards it. As we are interested in the outgoing radiation we solve Eq.(2.6) considering the radiation running to the cover (layer 1) and the substrate (layer p).

We assume that there is a horizontal (artificial) layer, say layer m, within the source layer containing the source as shown in figure (2.2). This layer has a very thin width d_m. We choose a local coordinate within each layer (except for the top layer), $x = 0 - d_m$, where d_m is the width of layer m. Considering the thin layer within the source layer, layer m, the x-dependent of delta function, $\delta(x - x_0)$, can be written as

$$\delta(x - x_0) = \lim_{d_m \to 0} \frac{1}{d_m} h(x - x_0), \tag{2.8}$$

where

$$h(x) = \begin{cases} 1 & \text{if } 0 < x < d_m \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

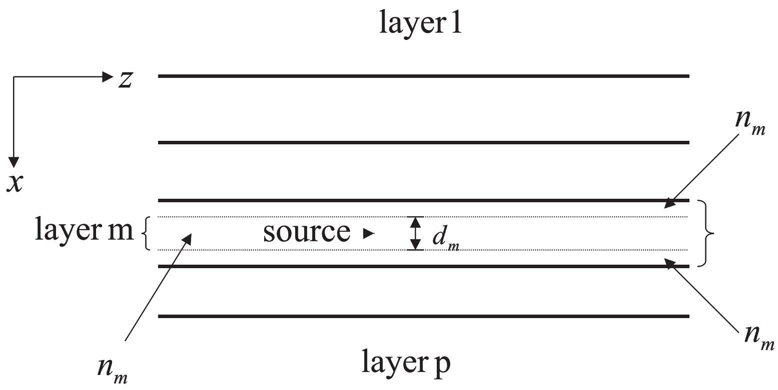

Figure 2.2: *Layered system with a horizontal layer m containing the source.*

By taking the Fourier transform of Eq.(2.6) with respect to z and by using $\delta(x-x_0) = \frac{1}{d_m}h(x-x_0)$ (before taking the limit: $\lim_{d_m \to 0}$) we obtain for each frequency k_z

$$[\partial_{xx} - k_z^2 + k_0^2 n^2(x)]G_r(k_z, x) = \frac{\omega^2 \mu_0}{d_m} G_0 h(x - x_0), \tag{2.10}$$

where

$$\begin{aligned} G_r(k_z, x) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} E_y(x, z) e^{ik_z z} dz, \\ G_0 &= \frac{1}{\sqrt{2\pi}} p_{2D,y} e^{ik_z z_0}. \end{aligned}$$

Eq. (2.10) can be solved for all k_z corresponding to plane waves running into the substrate and the cover.

Ansatz (1): The field solution in the source layer, layer m, is written as the sum of the homogeneous and inhomogeneous solutions as

$$G_{r,m}(k_z, x) = a_{m,+} e^{\alpha_m x} + a_{m,-} e^{-\alpha_m x} + b, \tag{2.11}$$

where b is a constant and $\alpha_m = \sqrt{k_z^2 - k_0^2 n_m^2}$, with $\mathrm{Re}(\alpha_m) > 0$ if $\mathrm{Re}(\alpha_m) \neq 0$, and $\mathrm{Im}(\alpha_m) > 0$ otherwise for real or imaginary values, respectively.

In this subsection we are only interested in the radiation fields. Thus we will only determine the field solutions for the imaginary values of α_m in the

outermost layer (for $m = 1$ and p).
By substituting Eq. (2.11) into Eq.(2.10) we obtain the inhomogeneous solution

$$b = \frac{\omega^2 \mu_0}{d_m} \frac{G_0}{\alpha_m^2}.$$

To determine the homogeneous solutions we solve Eq.(2.10) with ansatz (2.11) for outgoing waves in the substrate and cladding with the requirement that G_r and $\partial_x G_r$ are continuous along x. Using this fact expressions for the amplitudes of the outgoing plane waves in the cladding and substrate can be found for all relevant k_z values. The regions of the relevant propagation constant k_z of the cladding and the substrate is given by $|k_z| < K_1$ and $|k_z| < K_p$, respectively, where $K_q = k_0 n_q$, $q = 1, p$ and k_0 is the wave number in the free space.

The field solution (for given k_z) in layers 1 to $m-1$ should be identical to that corresponding to an incoming plane wave in layer m, leading also to an outgoing wave in layer 1. A similar reasoning holds for layers $m+1$ to p. Based on that, the ratio of the field and its derivative (with respect to x) should be equal to that of the corresponding (same k_z) field solution with outgoing fields in the outermost layers. Then we obtain [3]

$$\frac{\partial_x G_{r,l}}{G_{r,l}}|_{x=0} = \frac{\alpha_m(1 - r_{m,1})}{1 + r_{m,1}}, \tag{2.12}$$

$$\frac{\partial_x G_{r,m}}{G_{r,m}}|_{x=d_m} = \frac{\alpha_m(-1 + r_{m,p})}{1 + r_{m,p}}. \tag{2.13}$$

In the above r denote the Fresnel reflection coefficients, which correspond to the outgoing or evanescent waves in layer 1 and layer p, respectively.
The subscripts indicate the considered layer system, for example: $r_{m,p}$ is amplitude reflection for layers $p - m$.

Eqs.(2.12) and (2.13) can be solved for the two unknowns $(a_{m,\pm})$ (using Eq. (2.11) and before taking the limit $\lim_{d_m \to 0}$):

$$a_{m,+} = \frac{b}{D}[r_{m,p} e^{-\alpha_m d_m}(r_{m,1} - 1) - (1 - r_{m,p})], \tag{2.14}$$

$$a_{m,-} = \frac{b}{D}[e^{\alpha_m d_m}(r_{m,1} - 1) - r_{m,1}(1 - r_{m,p})], \tag{2.15}$$

where

$$D = 2(e^{\alpha_m d_m} - r_{m,1} r_{m,p} e^{-\alpha_m d_m}).$$

The outgoing fields in layer 1 can be determined using the standard reflection and transmission laws to relate the field in layer m to the outgoing field amplitude. Similar remark holds for the amplitudes of the outgoing field in layer p.

$$a_{1,+} = \frac{t_{m,1} G_{r,m}|_{x=0}}{1 + r_{m,p}}, \tag{2.16}$$

$$a_{p,-} = \frac{t_{m,p} G_{r,m}|_{x=d_m}}{1 + r_{1,p}}, \tag{2.17}$$

where t are the Fresnel transmission coefficients.
Substitute Eqs.(2.16) and (2.17) into Eqs. (2.14) and (2.15), and taking the limit $d \to 0$ we obtain

$$a_{1,+}^{TE} = \omega^2 \mu_0 \frac{1}{2\alpha_m} f_{1,+}(k_z) G_0, \tag{2.18}$$

$$a_{p,-}^{TE} = \omega^2 \mu_0 \frac{1}{2\alpha_m} f_{p,-}(k_z) G_0. \tag{2.19}$$

where

$$f_{1,+}(k_z) = \frac{t_{m,1}(1 + r_{m,p})}{(1 - r_{m,1} r_{m,p})}, \tag{2.20}$$

$$f_{p,-}(k_z) = \frac{t_{m,p}(1 + r_{m,1})}{(1 - r_{m,1} r_{m,p})}. \tag{2.21}$$

The field solutions of the radiation are found to be

$$E_y(k_z, x < x_1) = a_{1,+}^{TE} e^{\alpha_1 (x - x_1)}, \tag{2.22}$$

$$E_y(k_z, x > x_p) = a_{p,-}^{TE} e^{-\alpha_p (-x + x_p)}. \tag{2.23}$$

where the sign of α_1 and α_p is chosen similar to α_m (see Eq. (2.11)).
In the above we solve Eq. (2.6)for all k_z that lay in the regions $|k_z| < K_1$ and $|k_z| < K_p$ to determine the radiation fields. In the following we solve the same equation, Eq.(2.6) to determine the guided fields. Thus, we consider the field solutions corresponding to the real values of α_m in the outer most layer.

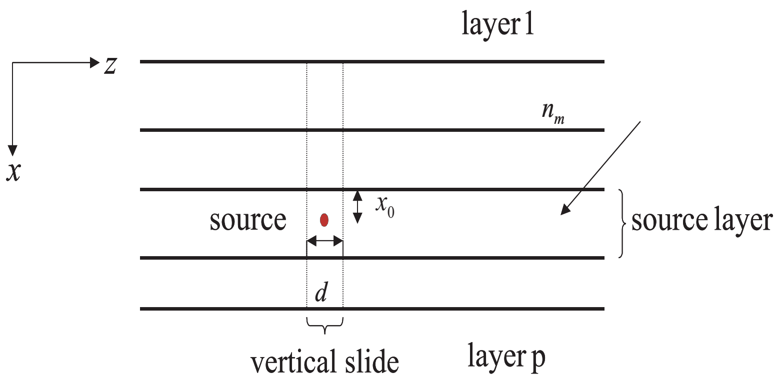

Figure 2.3: *Layered system with a vertical slide containing the source.*

2.2.2 Guided Fields

In this section we solve Eq.(2.6) to determine the amplitudes of the guided fields. Therefore, we only consider the fields solutions that are oscillating in the core layers and exponentially decaying in the outermost layers.
As shown in figure (2.3), this time we assume that the source is contained in a thin vertical slide of width d, so the z-dependent delta function can be expressed as

$$\delta(z - z_0) = \lim_{d \to 0} \frac{1}{d} l(z - z_0), \tag{2.24}$$

where

$$l(z) = \begin{cases} 1 & \text{if } \ 0 < z < d \\ 0 & \text{otherwise} \end{cases}$$

The x-dependent delta function term, $\delta(x - x_0)$, can be written as

$$\delta(x - x_0) = \sum_m a_m e_m(x) + f(x). \tag{2.25}$$

where $e_m(x)$ are the mode profiles. $f(x)$ represents the remaining part of the optical field and is orthogonal to $e_m(x)$ [7].
Assuming for the moment that the waveguide structure support only a single

guided mode, so we can find a_0 as

$$a_0 = \frac{e_0(x_0)}{\int\limits_{-\infty}^{\infty} [e_0(x)]^2\, dx}.$$

Higher order guided modes can be treated similarly.
We are looking for a solution with an x-dependence according to the zero-order mode. Thus, the field solution in term of the guided Field in the thin vertical slide $z = [0, d]$ is of the form

$$E_y(x,z) = [a_- e_0(x) e^{-i\beta z} + a_+ e_0(x) e^{i\beta z} + q e_0(x)] + g(x,z), \tag{2.26}$$

where β is the propagation constant of the guided modes and q is a constant. And with$g(x,z) \perp e_0(x)$ for the all values of z, because of the zero overlap between $e_0(x)$ and $g(x)$..
I In this subsection we are interested to determine the outgoing fields in the term of the guided fields. That means we will pay all our attention to determine the first part of the field.
The guided field for $z < 0$ is of the form $a_+ e_0(x) e^{i\beta z}$ and for $z > d$ is of the form $a_- e_0(x) e^{-i\beta z}$.
Substituting Eq. (2.26) into Eq.(2.6) and multiplying both sides by $e_0(x)$ it follows after integrating through with respect to x (x runs from $-\infty$ to $-\infty$)

$$q = \frac{\omega^2}{d\beta^2} \mu_0 p_{2D,y} a_0.$$

To determine the 4 other amplitudes, we apply the continuity conditions of the guided fields ,E_y and $\partial_z E_y$, at the interfaces $z = 0$ and $z = d$. The continuity conditions read [1]

$$E_y|_{z=0^-} = E_y|_{z=0^+}, \tag{2.27}$$
$$\partial_z E_y|_{z=0^-} = \partial_z E_y|_{z=0^+}, \tag{2.28}$$
$$E_y|_{z=d^-} = E_y|_{z=d^+}, \tag{2.29}$$
$$\partial_z E_y|_{z=d^-} = \partial_z E_y|_{z=d^+}. \tag{2.30}$$

We interested only in the amplitudes $a_\pm$. By applying Eqs. (2.27)-(2.30) we find

$$a_+ = a_- = \frac{-i}{2\beta} \omega^2 \mu_0 p_{2D,y} a_0.$$

Now the guided field can be written as

$$E_y(x,z) = \frac{-i}{2\beta}\omega^2 \mu_0 a_0 p_{2D,y} e_0(x) e^{-i\beta|z-z_0|}. \tag{2.31}$$

Other guided modes (if they are present) can be treated similarly. Then we will obtain similar expression as Eq. (2.31) with a superposition of the amplitudes of all present guided modes.

2.3 TM polarization

From Eq. (2.4) and by choosing the y component of the magnetic field, H_y, we can write the 2D wave equation for the TM polarization for the same structure, see figure (2.1), as

$$\begin{aligned}[n^2(x)\partial_x \frac{1}{n^2(x)}\partial_x + \partial_{zz} + k_0^2 n^2(x)]H_y(x,z) = \\ -i\omega\nabla \times (\vec{p}_{2D,j}\delta(x-x_0)\delta(z-z_0))|_y\end{aligned} \tag{2.32}$$

where the subscript $j = x, z$. The polarization of the radiating dipole a long y direction, $p_{2D,y}$, doesn't contribute to the TM polarization.
By taking into account the fact that

$$\nabla \times \delta(x-x_0)\delta(z-z_0) = -\nabla_0 \times \delta(x-x_0)\delta(z-z_0),$$

Eq.(2.32) can be written as

$$\begin{aligned}[n^2(x)\partial_x \frac{1}{n^2(x)}\partial_x + \partial_{zz} + k_0^2 n^2(x)]H_y(x,z) = \\ i\omega\nabla_0 \times (\vec{p}_{2D,j}\delta(x-x_0)\delta(z-z_0))|_y\end{aligned} \tag{2.33}$$

Now defining a potential $\vec{A}$ such that $H_y = \nabla_0 \times \vec{A}|_y$, then Eq. (2.33) becomes

$$\begin{aligned}[n^2(x)\partial_x \frac{1}{n^2(x)}\partial_x + \partial_{zz} + k_0^2 n^2(x)]\vec{A}_{x/z}(x,z) = \\ i\omega\vec{p}_{2D,j}\delta(x-x_0)\delta(z-z_0)\end{aligned} \tag{2.34}$$

where the subscript x/z indicate that the potential $\vec{A}$ lays on the $x-z$ plane. We are interested to calculate also the radiation and the guided fields for the TM polarization. To achieve this, we follow a similar reasoning as before for the TE polarization.

2.3.1 Radiation Fields

As before, first we consider the radiation fields. Considering to Eq. (2.34), in the following we calculate the radiation that is caused by a source that has a dipole along both x and z.
Similarly as in the previous section, the source term $\delta(x-x_0)$ can be expressed by Eq. (2.8). Applying this, and taking the Fourier transform of Eq. (2.34) with respect to z it follows for each spatial frequency k_z

$$[n^2(x)\partial_x \frac{1}{n^2(x)}\partial_x - k_z^2 + k_0^2 n^2(x)]\tilde{G}_r(k_z, x) = \frac{i\omega}{d_m}\tilde{G}_0(k_z)h(x-x_0), \quad (2.35)$$

where

$$\begin{aligned} \tilde{G}_r(k_z, x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_{x/z}(x, z) e^{ik_z z} dz, \\ \tilde{G}_0 &= \frac{1}{\sqrt{2\pi}} \vec{p}_{2D,j} e^{ik_z z_0}. \end{aligned}$$

As we mentioned in the previous section (from ansatz(1)), the solution in the source layer, layer m, can be written as the sum of the homogeneous and inhomogeneous solution as:

$$\tilde{G}_{r,m}(k_z, x) = \tilde{a}_{m,+}e^{\alpha_m x} + \tilde{a}_{m,-}e^{-\alpha_m x} + \tilde{b}. \quad (2.36)$$

The inhomogeneous solution can be determined directly by substituting Eq. (2.36) into Eq. (2.6)

$$\tilde{b} = \frac{i\omega}{d_m}\frac{\tilde{G}_0}{\alpha_m^2}$$

Since the G_r and $\partial_x G_r$ are continuous within the source layer Eqs. (2.12) and (2.13) are also hold in this case. Then applying the standard expressions for reflection and transmission , Eq.(2.16) and (2.17), the amplitudes of the outgoing fields in the outermost layers for Eq. (2.34) read

$$\begin{aligned} \tilde{a}_{1,+} &= i\omega \frac{t_{m,1}(1 + r_{m,p})}{2\alpha(1 - r_{m,1}r_{m,p})}\tilde{G}_0, & (2.37) \\ \tilde{a}_{p,-} &= i\omega \frac{t_{m,p}(1 + r_{m,1})}{2\alpha(1 - r_{m,1}r_{m,p})}\tilde{G}_0, & (2.38) \\ & & (2.39) \end{aligned}$$

But

$$\begin{aligned} a^{TM}_{1,+} &= \nabla_0 \times \tilde{a}_{1,+}|_y, \\ a^{TM}_{p,-} &= \nabla_0 \times \tilde{a}_{p,-}|_y. \end{aligned}$$

Then the amplitudes for the radiation fields are given by

$$a^{TM}_{1,+} = \frac{i\omega}{2\sqrt{\pi}\alpha}\left(ik_z p_{2D,x} f_{1,+}(k_z) + \alpha p_{2D,z} g_{1,+}(k_z)\right) e^{ik_z z_0}, \tag{2.40}$$

$$a^{TM}_{p,-} = \frac{i\omega}{2\sqrt{\pi}\alpha}\left(ik_z p_{2D,x} f_{p,-}(k_z) - \alpha p_{2D,z} g_{p,-}(k_z)\right) e^{ik_z z_0}, \tag{2.41}$$

where $f_{1,+}(k_z)$ and $f_{p,-}(k_z)$ are defined by Eqs. (2.20) and (2.21), respectively, and

$$g_{1,+}(k_z) = \frac{t_{m,1}(-1 + r_{m,p})}{(1 - r_{m,1} r_{m,p})}, \tag{2.42}$$

$$g_{p,-}(k_z) = \frac{t_{m,p}(-1 + r_{m,1})}{(1 - r_{m,1} r_{m,p})}. \tag{2.43}$$

Now the field solutions of the radiation can be expressed as

$$H_y(k_z, x < x_1) = a^{TM}_{1,+} e^{\alpha_1(x - x_1)}, \tag{2.44}$$

$$H_y(k_z, x < x_1) = a^{TM}_{p,-} e^{\alpha_p(-x + x_{p-1})}. \tag{2.45}$$

2.3.2 Guided Fields

In this section we solve Eq.(2.34) to find the amplitudes of the guided fields for the TM polarization. We follow a similar treatment as for the amplitudes of the guided fields of the TE polarization but in this case we treat twodimensional electric dipole source.

The z-dependent term of delta function, $\delta(z - z_0)$, can be represented by Eq. (2.24) and the x-dependence term can be written as

$$\delta(x - x_0) = \sum_m b_m \frac{h_m(x)}{n(x)} + f(x), \tag{2.46}$$

$$\tag{2.47}$$

where $h_m(x)$ are the guided mode profiles. $f(x)$ represents the remaining part of the optical field and is orthogonal to $h_m(x)/n(x)$ [7].

Consider, for the moment, the zero-order guided mode. Using Eq. (2.46), multiplying through by $h_0(x)/n^2(x)$ and integrating with respect to x, we determine b_0 as

$$b_0 = \frac{h_0(x_0)/n_m}{\int\limits_{-\infty}^{\infty} [h_0(x)/n(x)]^2 \, dx}, \tag{2.48}$$

$$\tag{2.49}$$

Following the same reasoning for the guided fields as in the previous section, the field solution in the vertical slide $z = [0, d]$ (see figure (2.3)) can be written as

$$A_{x/z}(x,z) = [\tilde{a}_- h_0(x) e^{-i\beta z} + \tilde{a}_+ h_0(x) e^{i\beta z} + \tilde{q} h_0(x)] + g(x,z). \tag{2.50}$$

where β is the propagation constant of the guided modes, $\tilde{q}$ is a constant and the subscript x/z indicates that A lays in the $x - z$ plane. And with $g(x,z) \perp h(x)/n(x)$ for all values of z.
As we mentioned before we are only interested to determine the guided field solutions. So we only need to determine the first part of the field.
The field solution of the guided mode for $z < 0$ is of the form $\tilde{a}_- h_0(x) e^{i\beta z}$ and for $z > d$ is of the form $\tilde{a}_+ h_0(x) e^{-i\beta z}$.
We can find the constant $\tilde{q}$ by substituting Eqs. (2.50)and (2.24)(before taking the limit) into Eq. (2.34), then multiplying through by $h_0(x)/n^2(x)$ it and integrating with respect to x (x runs from $-\infty$ to ∞):

$$\tilde{q} = \frac{i\omega}{d\beta^2} \frac{b_0}{n_m} \vec{p}_{2D,j}. \tag{2.51}$$

We apply the continuity conditions of H_y along z to determine the other 4 amplitudes. The continuity conditions read [1]

$$\begin{aligned} H_y|_{z=0^-} &= H_y|_{z=0^+}, & (2.52)\\ \partial_z H_y|_{z=0^-} &= \partial_z H_y|_{z=0^+}, & (2.53)\\ H_y|_{z=d^-} &= H_y|_{z=d^+}, & (2.54)\\ \partial_z H_y|_{z=d^-} &= \partial_z H_y|_{z=d^+}. & (2.55) \end{aligned}$$

We also aim to determine only $a_\pm$. Applying the interface conditions (2.52)-(2.55) we obtain after taking the limit $\lim_{d\to 0}$

$$\tilde{a}_+ = \tilde{a}_- = \frac{-\omega}{n_m} \frac{b_0}{2\beta} \vec{p}_{2D,j}.$$

The potential field solution can be written as

$$A_{x/z}(x,z) = \frac{-\omega}{2\beta}\frac{b_0}{n_m}\vec{p}_{2D,j}h_0(x)e^{-i\beta|z-z_0|}.$$

Now the magnetic field can be found by using $H_y(x,y) = \nabla_0 \times \vec{A}|_y$ as

$$H_y(x,y) = \frac{\omega}{2\beta n_m}h_0(x)\left[\pm i\beta p_{2D,x}b_0 + p_{2D,z}b_0^{'}\right]e^{-i\beta|z-z_0|}$$

where $b_0^{'}$ stands for $\frac{db_0(x_0)}{dx_0}$. And with the $+$ sign for $z > 0$ and the $-$ sign for $z < 0$
A similar expressions for the higher guided modes can be obtained.

Chapter 3

3D Problem

In the present chapter we extend the analytical expressions that have been derived for the radiation and the guided fields due to the radiating dipole in the core of a 2D dielectric layered structure to the 3D structure. We derive expressions for the radiated and the guided power. Here also both the TE and the TM polarizations are considered.

3.1 TE Polarization

The 3D wave equation for TE polarization is given by

$$[\triangle^2 + k_0^2 n^2(x)]\vec{E}(\vec{r}) = -\omega^2 \mu_0 \vec{p}\delta(\vec{r} - \vec{r}_0), \tag{3.1}$$

with electric field and dipole moment in the $y - z$ plane.
In this section we derive expressions for the power transferred to the radiation and the guided modes. First we solve Eq. (3.1) to determine the amplitudes of the radiation and the guided fields. To simplify the problem we use a rotation transform.

3.1.1 Rotation Transform

In order to cope with the complexity of the full vectorial equations (3D equations) we use a rotation transformation to reduce the 3D equations to the 2D form.
Then, by considering the analytical expressions of the radiation fields for the 2D formulation (see figure (2.1)), we can directly write the analytical

expression for the radiation fields for the 3D problem.
The polarization term for the 3D problem reads

$$\vec{P} = \vec{p}\delta(r - r_0) \tag{3.2}$$

where $\vec{r_0}$ is the position of the source,

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and

$$\vec{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

is the electric dipole momentum. Here the delta function is given by

$$\delta(\vec{r} - \vec{r_0}) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0). \tag{3.3}$$

The source term, Eq. (3.2), can be written in the two dimensional Fourier domain for given frequencies k_y and k_z as

$$\tilde{\vec{P}}(x, k_y, k_z) = \vec{p}\delta(x - x_0)\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-i(k_x(y-y_0)+k_z(z-z_0))}dk_y dk_z \tag{3.4}$$

In the uniform plane $y-z$, let the frequencies k_y, k_z be rotated anticlockwise by an angle θ. The rotated frequencies, which are denoted by $k_{\bar{y}}$ and $k_{\bar{z}}$, are given by

$$k_{\|} = \begin{pmatrix} k_{\bar{y}} \\ k_{\bar{z}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} k_y \\ k_z \end{pmatrix} \tag{3.5}$$

Similarly we can define $r_{\|}$ and $p_{\|}$ in the same plane.
In the rotated coordinates the polarization term can be written as

$$\vec{P}(x, y, z) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \tilde{\vec{P}} e^{ik_{\bar{z}}\bar{z}} dk_y dk_z. \tag{3.6}$$

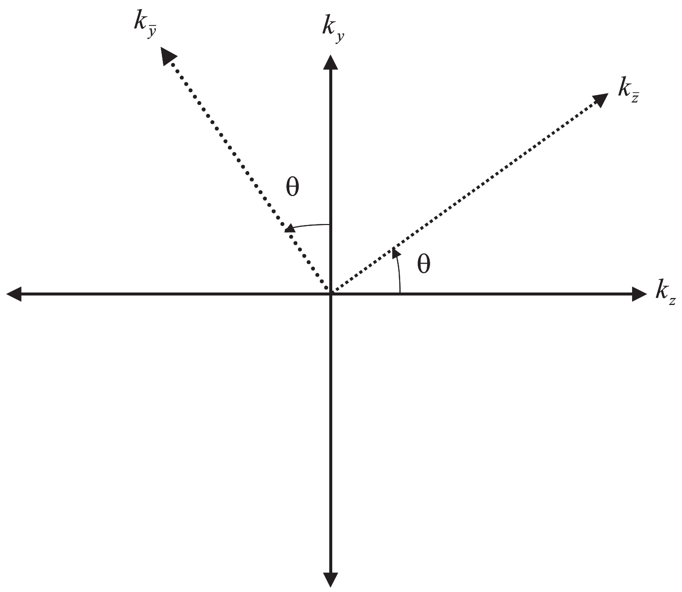

Figure 3.1: Rotation Transformation

The key point is that the polarization term $\vec{P}(x, k_y, k_z)$ corresponds to a field with x and y dependence according to

$$e^{ik_y(y-y_0)} + e^{ik_z(z-z_0)} = e^{ik_{\bar{z}}(\bar{z}-\bar{z}_0)},$$

where $\bar{z} \| k_{\|}$ $(\equiv k_{\bar{z}})$.
So, there is no $\bar{y}$ dependence which means $\partial_{\bar{y}} = 0$.
In this way the 3D equations for both the TE and TM polarizations can be reduced to the same form as the 2D equations.

3.1.2 Radiation Fields and Radiated Power

In this section we will derive expressions for the radiation Fields and the radiated power.
By applying the Fourier transform to Eq.(3.1) with respect to y and z we obtain for each frequencies k_y and k_z

$$[\partial_{xx} - (k_y^2 + k_z^2) + k_0^2 n^2(x)]\vec{\bar{E}}(x, k_y, k_z) = \frac{\omega^2 \mu_0}{d_m} \bar{G}_0 h(x), \tag{3.7}$$

where d_m is the width of layer m as shown in figure (2.1), h is defined by Eq. (2.9) and

$$\begin{aligned}\tilde{\vec{E}}(x,k_y,k_z) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\vec{E}(x,y,z)e^{ik_y(y-y_0)+ik_z(z-z_0)}dydz,\\ \bar{G}_0 &= \frac{1}{2\pi}\vec{p}e^{i(k_yy_0+k_zz_0)}.\end{aligned}$$

In the $y-z$ plane $\vec{E}\|\vec{P}$. Considering Eq. (3.6) and following similar reasoning for Eq. (2.10) the radiation modes in the outermost layers (layer 1 and layer p), respectively, read

$$\tilde{E}_{\bar{y}}(x<x_1,k_y,k_z) = \omega^2\mu_0\frac{f_{1,+}(k_\|)}{2\alpha_m}G_0e^{\alpha_1(x-x_1)}, \tag{3.8}$$

$$\tilde{E}_{\bar{y}}(x>x_p,k_y,k_z) = \omega^2\mu_0\frac{f_{p,-}(k_\|)}{2\alpha_m}G_0e^{\alpha_p(-x+x_{p-1})}. \tag{3.9}$$

where

$$G_0 = \frac{1}{2\pi}p_{\bar{y}}e^{ik_{\bar{z}}z_0},$$

$\bar{z}$ corresponds to $k_\|$. $f_{1,+}(k_\|)$ and $f_{p,-}(k_\|)$ are defined by Eqs. (2.20) and (2.21), respectively, $\alpha_m = \sqrt{(k_y^2+k_z^2)-k_0^2n_m^2}$, with $\mathrm{Re}(\alpha_m)>0$ if $\mathrm{Re}(\alpha_m)\neq 0$, and $\mathrm{Im}(\alpha_m)>0$ otherwise for real or imaginary values, respectively. As we mentioned in the previous chapter, for the radiation field we only consider the field solutions in the outermost layers that correspond to the imaginary values of α.

Using the Inverse Fourier transform the electric field in the outermost layer 1 and layer p, respectively, is given by

$$\tilde{E}_{\bar{y}}(x<x_1,y,z) = \int\limits_{-K_1}^{K_1}\int\limits_{-K_1}^{K_1}\tilde{E}_{\bar{y}}(k_y,k_z,x<x_1)dk_ydk_z, \tag{3.10}$$

$$\tilde{E}_{\bar{y}}(x>x_p,y,z) = \int\limits_{-K_p}^{K_p}\int\limits_{-K_p}^{K_p}\tilde{E}_{\bar{y}}(k_y,k_z,x>x_p)dk_ydk_z, \tag{3.11}$$

where the integration runs over the relevant k_y and k_z regions ($K_q = k_0n_q, q = 1,p$) which correspond to the fields radiated into layer1 and layer p.

In order to find the transmitted power we use the Poynting vector which reads

$$\vec{S} = \frac{1}{2} Re(\vec{E} \times \vec{H}^*). \tag{3.12}$$

where the subscript$*$ indicates the complex conjugate.
From Eq. (3.12) the radiated power along x into the outermost layers is given by

$$P_x = \frac{1}{2} Re \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} (\tilde{E}_y \tilde{H}_z^* - \tilde{E}_z \tilde{H}_y^*) dydz. \tag{3.13}$$

Thus, we further need to determine the corresponding magnetic fields. In this case it is only required to find the corresponding magnetic fields along $\bar{z}$ which follow from Maxwell's curl equations

$$\begin{aligned} \tilde{H}_{\bar{z}}(x < x_1, y, z) &= \frac{\alpha_1}{i\omega\mu_0} \tilde{E}(x < x_1, y, z), &(3.14)\\ \tilde{H}_{\bar{z}}(x > x_p, y, z) &= \frac{-\alpha_p}{i\omega\mu_0} \tilde{E}\bar{y}(x > x_p, y, z). &(3.15) \end{aligned}$$

Then, the radiated power into layer 1 can be determined by substituting Eqs. (3.10) and (3.14) into Eq. (3.13) as

$$\begin{aligned} P_{x,1} &= \frac{1}{8\pi^2} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} dydz \\ &\int\limits_{-K_1}^{K_1} \int\limits_{-K_1}^{K_1} \tilde{E}_{\bar{y}}(k_y, k_z, x < x_1) e^{-i(k_y(y-y_0)+k_z(z-z_0))} dk_y dk_z \\ &\int\limits_{-K_1}^{K_1} \int\limits_{-K_1}^{K_1} \tilde{H}_{\bar{z}}^*(k_y^{'}, k_z^{'}, x < x_1) e^{i(k_y^{'}(y-y_0)-k_z^{'}(z-z_0))} dk_y^{'} dk_z^{'} \end{aligned} \tag{3.16}$$

where the prime indicate that the involved parameter correspond to $\tilde{H}^*$.
Integrating over y, z and changing to the cylindrical coordinate system leads

to

$$
\begin{aligned}
P_{x,1} &= -\frac{\omega^3\mu_0}{32\pi^2}\int_0^{k_1}|\alpha_1|\frac{f_{1,+}^2}{|\alpha_m|^2}kdk \\
&\quad \int_0^{2\pi}(p_y^2\cos\theta + p_z^2\sin^2\theta - 2p_y\cos\theta 2p_z\sin\theta)d\theta, \qquad (3.17)\\
P_{x,1} &= -\frac{\omega^3\mu_0}{32\pi^2}(p_y^2+p_z^2)\int_0^{k_1}|\alpha_1|\frac{f_{1,+}^2}{|\alpha_m|^2}kdk \qquad (3.18)
\end{aligned}
$$

where

$$
\begin{aligned}
k_{\bar{z}} &= k_z\cos\theta,\\
k_{\bar{y}} &= k_y\sin\theta,\\
\theta &= \arctan\left(\frac{k_y}{k_z}\right),\\
k &= |k_y+k_z|.
\end{aligned}
$$

To simplify the integration, Eq. (3.17), we have used that

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{(-ik_y+ik_y')(z-z_0)}dk_y &= 2\pi\delta(k_y'-k_y),\\
\int_{-\infty}^{\infty} e^{(-ik_z+ik_z')(z-z_0)}dk_z &= 2\pi\delta(k_z'-k_z).
\end{aligned}
$$

Similarly an expression for the power radiated into layer p can be obtained as

$$
P_{x,p} = -\frac{\omega^3\mu_0}{32\pi^2}(p_y^2+p_z^2)\int_0^{K_p}|\alpha_p|\frac{f_{p,-}^2(k_\parallel)}{|\alpha_m|^2}kdk \qquad (3.19)
$$

3.1.3 Guided Fields and Guided Power

Now we solve Eq. (3.1) to determine the guided fields and to derive expression for the transmitted power along the z direction. As in the previous chapter we

will also assume that the waveguide structure supports only a single guided mode which has an amplitude a_0.
Since we are looking for a solution with an x-dependence according to the zero-order mode, so we can define the electric field $\vec{E}(x,y,z)$ as

$$\vec{E}(x,y,z) = e_0(x)\vec{E}_h(y,z) + \vec{g}(x,y,z). \tag{3.20}$$

with $g(x,y,z) \perp e_0(x)$ for all values of y and z, because of the zero overlap between $e_0(x)$ and $g(x)$. [7].
Here we aim to find outgoing fields in the term of the guided fields. Thus, we will pay all our attention to determine the first part of the field.
By substituting Eqs.(3.20), (2.24) and (2.25) into Eq. (3.1) we remove the x-dependence in both side of the equation after integrating through with respect to x (x runs from $-\infty$ to ∞) and obtain

$$(\partial_{yy} + \partial_{zz} + \beta^2)\vec{E}_h(y,z) = -\omega^2\mu_0 a_0 \vec{p}\delta(y-y_0)\delta(z-z_0) \tag{3.21}$$

where β is the propagation constant of the guided mode $e_0(x)$.

As $\vec{E}_h(y,z)$ lies in the $y-z$ plane (which is a uniform domain) the corresponding problem, Eq. (3.21) corresponds to a 2D problem in a system with refractive index $N_0 = \beta/k_0$, so corresponding equation using Maxwell equations may be written as

$$[\partial_{yy} + \partial_{zz} + \beta^2]H_{x,h}(y,z) = -\omega^2\mu_0 a_0 \nabla \times [\vec{p}\delta(y-y_0)\delta(z-z_0)]|_x \tag{3.22}$$

The solution of Eq. (3.22) according to Eqs. (2.40) and (2.41) is given by

$$\tilde{H}_{x,h}(k_y, z<z_0) = \frac{\omega^2\mu_0 a_0}{2\sqrt{2\pi}ik_z}\left(ik_y k_z + ik_z p_y\right)e^{ik_y y_0 + ik_z(z-z_0)}, \tag{3.23}$$

$$\tilde{H}_{x,h}(k_y, z>z_0) = \frac{\omega^2\mu_0 a_0}{2\sqrt{2\pi}ik_z}\left(k_y p_z - ik_z p_y\right)e^{ik_y y_0 - ik_z(z+z_0)}. \tag{3.24}$$

From Eqs. (3.23) and (3.20) and curl Maxwell equations we can now express the corresponding electric field as

$$\tilde{E}(k_y,x,z) = \frac{e_0(x)}{i\omega\varepsilon_0 N_0^2}\nabla \times \tilde{H_{x,h}}(k_y,z), \tag{3.25}$$

It follows from Eq. (3.25)

$$\tilde{E}_y(k_y, x, z) = \frac{1}{i\omega\varepsilon_0 N_0^2} e_0(x) i k_z \tilde{H_{x,h}}(k_y, z),$$
$$\tilde{E}_z(k_y, x, z) = \frac{1}{i\omega\varepsilon_0 N_0^2} e_0(x) i k_y \tilde{H_{x,h}}(k_y, z).$$

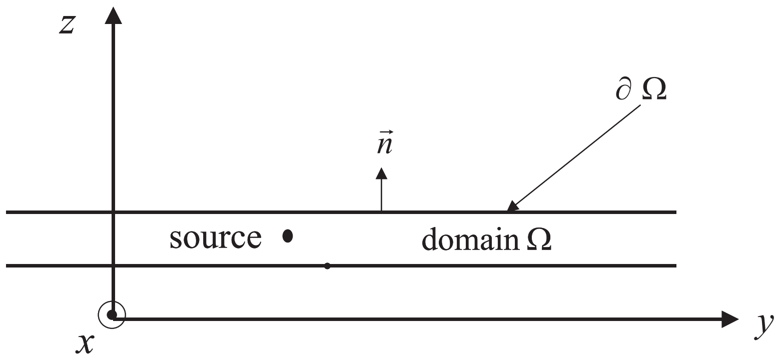

Figure 3.2: *The integration domain of the transmitted power along z direction*

The total power that transmitted along z direction is given by

$$\begin{aligned} P_z &= \int\limits_{\partial\Omega} \vec{S}_z(\vec{r}) \cdot \vec{n}(\vec{r}) dA \\ &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \vec{S}_z(x, y, z_0) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dxdy \\ &+ \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \vec{S}_z(x, y, z_0) \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dxdy \end{aligned} \tag{3.26}$$

where $\vec{S}(\vec{r})$ is the Poynting vector, $\vec{n}(\vec{r})$ is the normal vector and $\partial\Omega$ is the boundary of the domain Ω as show in figure (3.2).
According to the Eq(3.26) the guided power along z is given by

$$P_z = \frac{1}{2} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} (\vec{E}_x \vec{H}_y^* - \vec{E}_y \vec{H}_x^*) dxdy|_z \tag{3.27}$$

Therefore, for the power transmitted into, say the negative z direction, we only need to calculate the x-component of the magnetic field. Using Maxwell equations and Eq. (3.25) we readily find

$$\tilde{H_x}(k_y, x, z) = \frac{1}{\omega^2 \varepsilon_0 \mu_o N_0^2}(k_z^2 + k_y^2) e_0(x) \tilde{H_{x,h}}(k_y, z) = e_0(x) \tilde{H_{x,h}}(k_y, z). \quad (3.28)$$

We have used the facts that

$$\begin{aligned} \frac{1}{c^2} &= \varepsilon_0 \mu_0, \\ k_0^2 &= \frac{\omega^2}{c^2}, \\ \beta^2 &= k_0^2 N_0^2. \end{aligned}$$

where c is the speed of light in the free space.

The guided power in the $-z$ direction follows from Eq. (3.27)

$$\begin{aligned} P_{-z} &= \frac{\omega a_0^2}{8\varepsilon_0 N_0} Re \int_{-\infty}^{\infty} e_0^2(x) dx \int_{-\beta}^{\beta} \frac{1}{k_z}(k_y p_z + k_z p_y)^2 dk_y, && (3.29) \\ &= \frac{\omega a_0^2 k_0^2}{16\varepsilon_0}(\pi p_y^2 + \pi p_z^2 + 4 p_y p_z) \int_{-\infty}^{\infty} e_0^2(x) dx, && (3.30) \end{aligned}$$

where we have used

$$\begin{aligned} \int_0^{\beta} \frac{k_y^2}{k_z} dk_y &= \frac{\pi \beta^2}{4}, \\ \int_0^{\beta} k_z k_y dk_y &= \frac{\pi \beta^2}{4}, \\ \int_0^{\beta} 2 k_y dk_y &= \beta^2. \end{aligned}$$

A similar expression can be obtained for the guided power along positive z direction as

$$P_{+z} = \frac{\omega a_0^2 k_0^2}{16\varepsilon_0}(\pi p_y^2 + \pi p_z^2 - 4p_y p_z) \int_{-\infty}^{\infty} e_0^2(x)dx, \tag{3.31}$$

Now we can write the total power that is transmitted along z as

$$P_{g,\mathrm{tot}}^{TE} = \frac{\omega a_0^2 k_0^2 \pi}{16\varepsilon_0}(p_y^2 + p_z^2) \int_{-\infty}^{\infty} e_0^2(x)dx, \tag{3.32}$$

where the subscript indicate the total guided power.

3.2 TM Polarization

In this section we follow a similar reasoning as for the TE polarization to find analytical expressions for the radiated and the guided power.
The 3D wave equation for TM polarization is given by

$$[n^2(x)\partial_x \frac{1}{n^2(x)}\partial_x + \partial_{yy} + \partial_{zz} + k_0^2 n^2(x)]\vec{H}(\vec{r}) = -i\omega\nabla \times [\vec{p}\delta(\vec{r} - \vec{r}_0)], \tag{3.33}$$

Using the fact that $\nabla \times \delta(\vec{r} - \vec{r_0}) = -\nabla_0 \times \delta(\vec{r} - \vec{r_0})$ Eq.(3.33) can be written as

$$[n^2(x)\partial_x \frac{1}{n^2(x)}\partial_x + \partial_{yy} + \partial_{zz} + k_0^2 n^2(x)]\vec{H}(\vec{r}) = -i\omega\nabla \times_0 [\vec{p}\delta(\vec{r} - \vec{r}_0)], \tag{3.34}$$

where ∇_0 indicates that the first order derivatives are taken with respect to x_0, y_0 and z_0.
It should be noted that for the TM polarization all components of the dipole moment contribute.

3.2.1 Radiation Fields and Radiated Power

In this subsection we will derive expressions for the radiation Fields and the radiated power in the case of the TM polarization.

First we get rid of the curl that is present in the right hand side of Eq. (3.34) by introducing a potential $\vec{A}(\vec{r})$ such that

$$H(\vec{r}) = \nabla_0 \times \vec{A}(\vec{r}). \tag{3.35}$$

By substituting Eq.(3.35) into Eq. (3.34) we obtain

$$[n^2(x)\partial_x \frac{1}{n^2(x)}\partial_x + \partial_{yy} + \partial_{zz} + k_0^2 n^2(x)]\vec{A}(\vec{r}) = i\omega\vec{p}\delta(\vec{r} - \vec{r}_0), \tag{3.36}$$

Now taking the Fourier transform of Eq (3.36) with respect to y and z it follows

$$[n^2(x)\partial_x \frac{1}{n^2(x)}\partial_x - (k_y^2 + k_z^2) + k_0^2 n^2(x)]\vec{\tilde{A}}(x, k_y, k_z) = \frac{i\omega}{d_m}\bar{G}_0 h(x), \tag{3.37}$$

where

$$\begin{aligned} \vec{\tilde{A}}(x, k_y, k_z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{A}(\vec{r})e^{ik_y + ik_z}dydz, \\ \bar{G}_0 &= \frac{1}{2\pi}\vec{p}e^{ik_y y_0 + ik_z z_0}. \end{aligned}$$

By considering the rotation transform in the Fourier domain, Eq. (3.6), and following the same steps as for Eq. (3.1) we obtain

$$\vec{\tilde{A}}_{x/\bar{z}}(x < x_1, k_y, k_z) = \frac{-i\omega}{2\pi}\alpha_1 f_{1,+}(k_\parallel)G_0 e^{\alpha_m(x - x_1)}, \tag{3.38}$$

$$\vec{\tilde{A}}_{x/\bar{z}}(x > x_p, k_y, k_z) = \frac{-i\omega}{2\pi}\alpha_p f_{p,-}(k_\parallel)G_0 e^{\alpha_m(-x + x_{p-1})} \tag{3.39}$$

where

$$G_0 = \frac{1}{2\pi}\vec{p}_j e^{ik_{\bar{z}}\bar{z}_0}, \quad j = x, \bar{z}$$

From Eq. (3.35) the radiation fields in the outermost layers (layer 1 and layer p) can be obtained as

$$\begin{aligned} \vec{\tilde{H}}_{\bar{y}}(x < x_1, k_y, k_z) &= \frac{i\omega}{2\pi\alpha_1}\left(ik_{\bar{z}}p_x f_{1,+}(k_\parallel) + \alpha_m p_{\bar{z}} g_{1,+}(k_\parallel)\right) \\ &\quad G_0 e^{\alpha_m(x - x_1)}, \end{aligned} \tag{3.40}$$

$$\begin{aligned} \vec{\tilde{H}}_{\bar{y}}(x > x_p, k_y, k_z) &= \frac{i\omega}{2\pi\alpha_p}\left(ik_{\bar{z}}p_x f_{p,-}(k_\parallel) - \alpha_m p_{\bar{z}} g_{p,-}(k_\parallel)\right) \\ &\quad G_0 e^{\alpha_m(-x + x_{p-1})}. \end{aligned} \tag{3.41}$$

In order to express the power radiated into layer 1 we further need to find the corresponding electric field which follows from Maxwell curl equations as

$$\vec{\tilde{E}}_{\bar{z}}(x < x_1, k_y, k_z) = \frac{1}{-i\omega\varepsilon_0 n_1^2}\alpha_1 \tilde{H}_{\bar{y}}(x < x_1, k_y, k_z), \tag{3.42}$$

$$\vec{\tilde{E}}_{\bar{z}}(x > x_p, k_y, k_z) = \frac{1}{-i\omega\varepsilon_0 n_p^2}\alpha_p \tilde{H}_{\bar{y}}(x < x_1, k_y, k_z). \tag{3.43}$$

Then from Eq. (3.13) the radiated power into layer 1 in the cylindrical coordinate system can be expressed as

$$P_{x,1} = \frac{i\omega}{32\pi^2\varepsilon_0 n_1^2}\int_0^{k_1}\int_0^{2\pi} |\alpha_1| k_{\bar{z}} p_x \frac{f_{1,+}(k_\parallel)}{\alpha_m} + p_{\bar{z}} g_{1,+}(k_\parallel)|^2 k d\theta dk \tag{3.44}$$

$$P_{x,1} = \frac{i\omega}{32\pi\varepsilon_0 n_1^2}\int_0^{k_1} \alpha_1 \left[2\left(p_x(k_y^2+k_z^2)\frac{|f_{1,+}(k_\parallel)|}{\alpha_m}\right)^2 + |g_{1,+}(k_\parallel)|^2(p_y^2+p_z^2)\right] kdk \tag{3.45}$$

A similar expression can be obtained for the power radiated into layer p as

$$P_{x,p} = \frac{i\omega}{32\pi\varepsilon_0 n_p^2}\int_0^{k_p} \alpha_p \left[2\left(p_x(k_y^2+k_z^2)\frac{|f_{p,-}(k_\parallel)|}{\alpha_m}\right)^2 + |g_{p,-}(k_\parallel)|^2(p_y^2+p_z^2)\right] kdk \tag{3.46}$$

3.2.2 Guided Fields and Guided Power

In this section we solve Eq. (3.36) to find the expressions for the guided fields and the guided power for the TM polarization. We also assume that only a single guided mode is present with amplitude b_0. As we mentioned before the dipole moment is present in all directions, x, y and z.

Instead of solving Eq. (3.34) we solve Eq. (3.36) to avoid the complexity that is caused by the curl term that is present in the former equation.
We are looking for a solution with an x dependence according to the zero-order mode, so we can define the potential $\vec{A}(\vec{r})$ as

$$A(\vec{r}) = h_0(x)A_h(y,z) + g(x,y,z). \tag{3.47}$$

with $g(x,y,z) \perp h_0(x)/n(x)$ for the all values of y and z [7].
In the present subsection we are interested to find the outgoing fields in the

term of the guided fields. Thus, we will pay all out attention to determine the first part of the field.
From Eq.(3.35) it follows that

$$h_0(x)\vec{H}_h(y,z) = h_0(x)\nabla_0 \times \vec{A}_h(y,z). \tag{3.48}$$

Substituting Eq. (3.47) into Eq. (3.36) and writing $\delta(x-x_0) = b_0 h_0(x)$ (before taking the limit $\lim_{d\to 0}$) then, multiplying through by $h_0(x)/n^2(x)$ it follows after integrating through with respect to x (x runs from $-\infty$ to ∞)

$$[\partial_{yy} + \partial_{zz} + \beta^2]\vec{A}_h(y,z) = i\omega\vec{p}b_0\delta(y-y_0)\delta(z-z_0) \tag{3.49}$$

where $\beta = \sqrt{k_y^2 + k_z^2}$ is the propagation constant of the zero-order guided mode $h_0(x)$.
Eq. (3.49) corresponds to 2D problem in a uniform system with refractive index $N_0 = \beta/k_0$ with E along x. First we solve Eq. (3.49) to find $\vec{A}_h(y,z)$ then using Eq. (3.48) we determine all components of $\vec{H}_h$. Once the $\tilde{H}_h$ are found we apply the Maxwell curl equations to determine $\tilde{E}_{h,x}$.
From Eq.(2.22) in the previous chapter, the solution of Eq. (3.49) is found as

$$\tilde{A}_h(k_y, z < z_0) = p_j b_0 \gamma, \tag{3.50}$$

where

$$\gamma = \frac{\omega}{2\sqrt{2\pi}k_z} e^{i(k_y y_0 + k_z z_0(z-z_0))}. \tag{3.51}$$

The components of $\tilde{\vec{H}}_h(k_y,z)$ are determined by taking the curl_0 of Eq. (3.50)

$$\begin{aligned}
\tilde{\vec{H}}_{h,x}(k_y, z < z_0) &= \gamma b_0\left(-ik_y p_z + ik_z p_y\right) & (3.52)\\
\tilde{\vec{H}}_{h,y}(k_y, z < z_0) &= -\gamma\left(ik_z b_0 p_x + b_0' p_z\right) & (3.53)\\
\tilde{\vec{H}}_{h,z}(k_y, z < z_0) &= \gamma\left(b_0' p_y - ik_y b_0 p_z\right) & (3.54)
\end{aligned}$$

where b_0' stands for $\dfrac{db_0}{dx_0}$.
Using the curl Maxwell equation we can now express the electric field $\tilde{E}_{h,x}$ as

$$\tilde{E}_{h,x}(k_y, z < z_0) = \frac{\gamma}{i\omega\varepsilon_0 N_0^2}\left[\beta^2 b_0 p_x + ik_y b_0' p_y - ik_z b_0' p_z\right]. \tag{3.55}$$

From the Maxwell curl equations and Eq.(3.48) the resulting magnetic fields read

$$\tilde{\vec{H}}_y(x,k_y,z<z_0) = \frac{ik_z}{i\omega\mu_0}h_0(x)\tilde{E}_{h,x}(k_y,z<z_0), \tag{3.56}$$

$$\tilde{\vec{H}}_z(x,k_y,z<z_0) = -\frac{ik_y}{i\omega\mu_0}h_0(x)\tilde{E}_{h,x}(k_y,z<z_0). \tag{3.57}$$

A similar expressions can be obtained for the magnetic fields in the positive y direction.
Now we apply Eq. (3.26) to calculate the total power. As we mentioned in the previous section it is sufficient to calculate the power either along y or along z direction. For the power transmitted say into negative z direction we only need to determine the x-component of the electric field. Thus, by using the Maxwell curl equations we readily find

$$\tilde{\vec{E}}_x(x,k_y,z<z_0) = \frac{\gamma}{\omega^2\varepsilon_0\mu_0 n^2(x)}\beta^2\tilde{E}_{h,x}(k_y,z<z_0). \tag{3.58}$$

From Eq. (3.27) the total power guided along the negative z direction is given by

$$P_{-z} = \frac{1}{2\pi}Re\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}dxdz\int\limits_{\beta}^{-\beta}\tilde{\vec{E}}_x(x,k_y,z<z_0)dk_z\int\limits_{\beta}^{-\beta}\tilde{\vec{H}}_y^*(x,k_y^{'},z<z_0)dk_z^{'} \tag{3.59}$$

where the subscript $'$ indicates that the involved component corresponds to $\tilde{\vec{H}}^*$.
Performing the integration and using the fact that along positive z the same amount of power is running, it follows

$$P_{\text{g,tot}}^{TM} = \frac{w}{16n_m^2\varepsilon_0\int\limits_{-\infty}^{\infty}\frac{h_0^2(x)}{n^2(x)}}\left[2\beta^2h_0^2(x_0)p_x^2 + h_0^{'2}(x_0)(p_y^2+p_z^2)\right]. \tag{3.60}$$

Chapter 4

Discussion of the Results

Although our method is different for that of Lukosz [8] (see also [4]) our result expressions for the power form a radiating dipole going to the radiation modes agrees with those of [8]. In the present chapter we will focus mainly on layered structures supporting also a guided modes in order to illustrate our theory and also to show that the obtained results make sense, thus supporting the correctness of the presented theory for application for structures with guided modes.

4.1 Uniform Space

As we present in the following the radiated power (by a dipole in a layered system with refractive index $n(x)$) relative to that for a uniform space (with refractive index n) we will first give expressions for the later case.
We assume a uniform space with refractive index n.The total radiated power in the uniform space for the TE polarization is given by, according to Eqs. (3.18) and (3.19)

$$P_0^{TE} = \frac{\omega^3 \mu_0}{16\pi}(p_y^2 + p_z^2) \int\limits_0^K \frac{k}{\alpha} dk = \frac{\omega^3 \mu_0}{16\pi} K(p_y^2 + p_z^2), \tag{4.1}$$

where $K = k_0 n$ and $\alpha = \alpha_j$, $j = 1, p, m$.
The total radiated power in the case of the TM polarization follows from

Eqs. (3.45) and (3.46)

$$\begin{aligned}
P_0^{TM} &= \frac{\omega}{16\pi\varepsilon_0 n^2}\left[p_x^2 \int_0^K 2\frac{k^3}{\alpha}dk + (p_y^2+p_z^2)\int_0^K \alpha k dk\right] \\
&= \frac{\omega K^3}{48\pi\varepsilon_0 n^2}(4p_x^2+p_y^2+p_z^2), \\
P_0^{TM} &= \frac{\omega^3 K \mu_0}{48\pi}(4p_x^2+p_y^2+p_z^2). \qquad (4.2)
\end{aligned}$$

We have used that

$$k_0^2 = \frac{\omega^2}{c^2},$$

$$c^2 = \frac{1}{\varepsilon_0\mu_0}.$$

4.2 System closed with metal/manetic walls

Here we calculate the total power in a system closed with a metal walls and magnetic walls for the TE and the TM polarization, respectively.

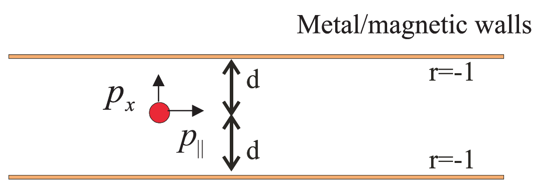

Figure 4.1: *Radiating dipole at the origin of a system closed with metal/magnetic walls*

For the TE polarization, we consider a source at the origin of a system closed with metal walls at $\pm d$, where $2d$ is the width of the waveguide, as shown in figure (4.1). In such case the eignfunctions (guided modes) are given by [1]

$$e_m(x) = \sin\{k_{x,m}(x-d)\}, \qquad (4.3)$$

Thus,

$$\begin{aligned} e_{2m-1}(0) &= 1, & (4.4) \\ e_{2m}(0) &= 0. & (4.5) \end{aligned}$$

where $k_{m,x} = m\pi/2d$ and the subscript m indicates the order of the considered mode.
From Eqs. (4.4) and (4.5) it can be seen that only the power that is transferred to the odd guided modes contributes. Applying Eq. (3.32) it follows that the power that is transmitted to each odd mode

$$P^{TE}_{\text{2m-1}} = \frac{\omega k_0^2}{16 d\varepsilon}(p_y^2 + p_z^2), \tag{4.6}$$

where $d = \int\limits_{-\infty}^{\infty} e_m^2(x)dx$.
The spacing between the odd modes is $\Delta k_x = \pi/d$, with k_x in the region $(0, k_0 n)$. For large d-values the summation over all modes leads to

$$\begin{aligned} P^{TE}_{\text{tot}} &= \frac{\omega k_0^2}{16 d\varepsilon}(p_y^2 + p_z^2) \times \frac{k_0 n d}{\pi}, \\ P^{TE}_{\text{tot}} &= \frac{\omega^3 K \mu_0}{16\pi}(p_y^2 + p_z^2). & (4.7) \end{aligned}$$

And so for TM polarization, we consider a source at the origin of a system closed with magnetic walls at $\pm d$ as shown in figure 4.1, where $2d$ is the width of the waveguide.

The eigenfuntions of the system are given by Eqs (4.3)-(4.5). For the sake of simplicity we will treat the contribution of the polarization along x direction, p_x^2, and the contribution of the polarization on the $y - z$ plane, $p_{\|}^2 (= p_y^2 + p_z^2)$ separately.
First we treat the contribution of a dipole oscillating along x direction. From Eq. (3.60) the guided power that is transferred by an odd guided mode m is given by

$$P^{TM}_{2m-1,p_x} = \frac{\omega}{8 d\varepsilon_0 n^4}\beta_m^2 p_x^2, \tag{4.8}$$

where $\beta_m^2 = \sqrt{k_0^2 n - k_{x,m}^2}$.

The total power can be written as

$$\begin{aligned}
P^{TM}_{tot,p_x} &= \sum_{m=0}^{k_0} \frac{\omega}{8d\varepsilon_0}\beta_m^2 p_x^2 \times \frac{d}{\pi}\Delta k_x, \\
&= \sum_{m=0}^{k_0} \frac{\omega}{8\pi\varepsilon_0 n^4} p_x^2 \int_0^{k_0 n^2} \beta_m^2 dk_x, \\
P^{TM}_{\text{tot},p_x} &= \frac{\omega^3 K\mu_0}{12\pi}\beta_m^2 p_x^2. \qquad (4.9)
\end{aligned}$$

Similarly we can calculate the contribution of a dipole oscillating in the $y-z$ plane. From Eq. (3.60) it follows

$$P^{TM}_{2m-1,p_\parallel} = \frac{k_0^4 k_{x,m}^2}{16d\omega^3\varepsilon_0\mu_0 n^4} p_\parallel^2, \qquad (4.10)$$

then it follows for the total power

$$\begin{aligned}
P^{TM}_{\text{tot},p_\parallel} &= \frac{k_0^4 k_{x,m}^2}{16d\omega^3\varepsilon_0\mu_0 n^4} p_\parallel^2 \times \frac{d}{\pi} \int_0^{k_0 n} k_x^2 dkx, \\
P^{TM}_{\text{tot},p_\parallel} &= \frac{\omega^3 K\mu_0}{48\pi}(p_y^2 + p_z^2). \qquad (4.11)
\end{aligned}$$

From Eqs. (4.9) and (4.11) the total power for the TM polarization is given by

$$P^{TM}_{\text{tot}} = \frac{\omega^3 K\mu_0}{48\pi}(4p_x^2 + p_y^2 + p_z^2). \qquad (4.12)$$

For both the TE and the TM polarization, Eqs. (4.1), (4.2), (4.7) and (4.12) show that the total power that radiated into a uniform space is exactly equal to the total power that is transmitted to the guided modes in a system closed with metallic/magnetic walls as it should.

In the following we discuss the obtained result for the total power that is transmitted in a uniform space (or a system closed with metal/magnetic walls).

	p_x	$p_\parallel$
R_{TE}	0	3/4
R_{TM}	1	1/4
R_{tot}	1	1

Table 4.1: *Relative power, $R = P/P_{tot,uniform}$, for a single radiating dipole*

In the following we use the radiative power relative to that in a uniform space with refractive index equal to the material surrounding the dipole $R_{TE} = P_{TE}/P_{\text{tot,uniform}}$ and $R_{TM} = P_{TE}/P_{\text{tot,unifrom}}$ for testing and analyzing the obtained expressions.
As we can see in the last row of table (4.1), the total relative power for both the TE and the TM is equal to 1. This means that the total radiated power of a dipole in a uniform space does not depend on the orientation of the radiating dipole as expected.

	p_x	$p_\parallel$	$\sum R$
R_{TE}	1×0	$2 \times 3/4$	1.5
R_{TM}	1×1	$2 \times 1/4$	1.5

Table 4.2: *Summation of relative power in a uniform space, $R = P/P_{tot,uniform}$, due to a single radiating dipole*

In the last column of table (4.2) we can see that , the summation of the relative power in a uniform space over p_x, p_y and p_z for both the TE and TM polarization are equivalent. Which means that for a large number of radiating oriented dipoles the power going into the TE polarization is equal to that going into the TM polarization, as expected.

4.3 Slab Wave Guides

In this section is to apply the obtained result to simple layered structures. In the following we use a relative power summed over the TE the TM polarization, $R_{\text{guided}} = P_{\text{guided}}/P_{\text{tot,uniform}}$ and $R_{\text{radiated}} = P_{\text{radiated}}/P_{\text{tot,uniform}}$. Matlab simulation environment version 6.5 is used for the programming. The required input for the main program are the wavelength λ, the thicknesses t and refractive indexes n of the all layers. Using these input values we can determine the profile for the relative radiated power and the relative guided power for the TE and the TM polarization and for all dipole orientations.

- **Example 1**

  As a first example we calculate (simulate) the total relative power in a three layered symmetric system with a radiating dipole positioned at the center of the core layer as show in figure(4.2). We present the

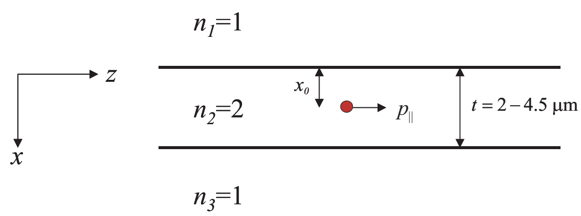

Figure 4.2: *Radiating dipole at the origin of a three layered system.*

  result for the relative power as a function of the thickness t of the core layer as shown in figure(4.3). We observe from the figure that the total relative power oscillates around 1 with an amplitude that gets smaller for large values of t. This simple example support the validity of the obtained expressions.

- **Example 2**

  Here we calculate the relative total power as a function of the position x_0. We apply the expressions of the total power to a three layered planar structures that support different number of modes.
  In the first case we take a structure that supports a single guided mode. Then we increase the number of modes one by one, for both polarizations, by increasing the refractive index of the core layer, see figure (4.4).

  In the first case we consider a structure with a fundamental mode. We find that, as shown in figure (4.5), the guided power monotonically

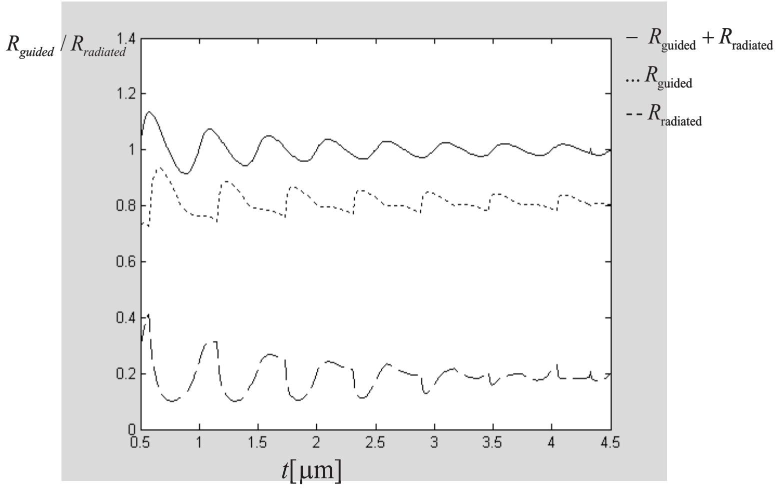

Figure 4.3: *Relative power summed over the TE and TM polarization form a radiating dipole in the core of symmetric slab waveguide.*

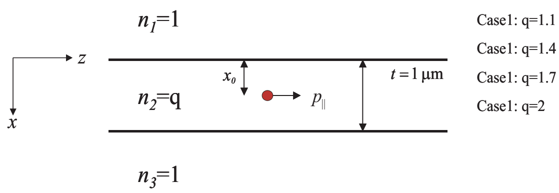

Figure 4.4: *Symmetric slab waveguide*

increasing towards the center of the core layer. That is reasonable because the highest value of the fundamental modes positioned at the center of the core layer.

As shown in figures (4.6)-(4.8) for the other cases it found that the

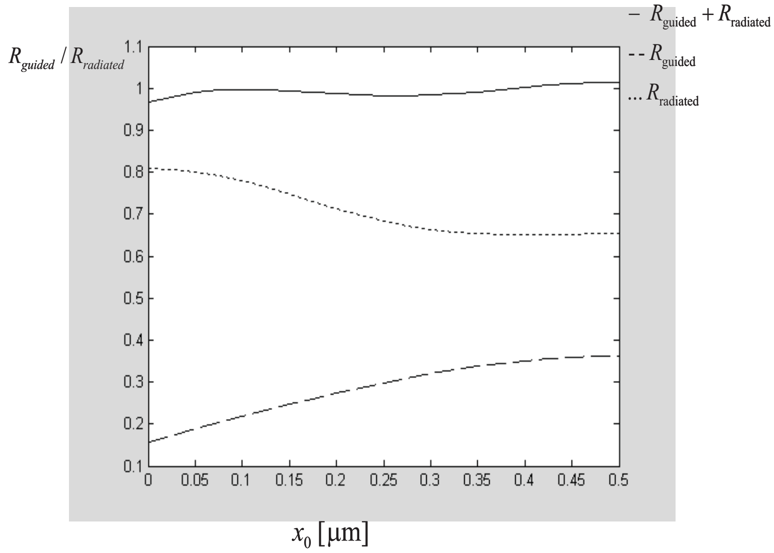

Figure 4.5: *case1: Single guided mode*

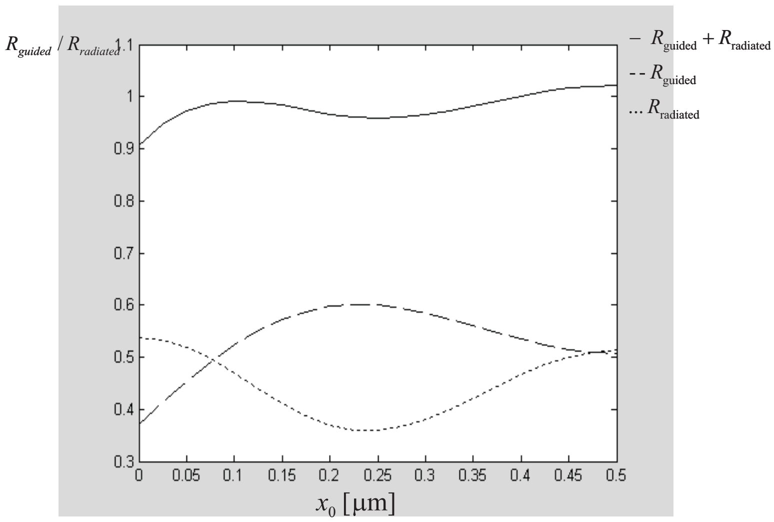

Figure 4.6: *case2:Two guided modes*

total guided power oscillating periodically according to the number of the guided modes that are found in each structure. These results also

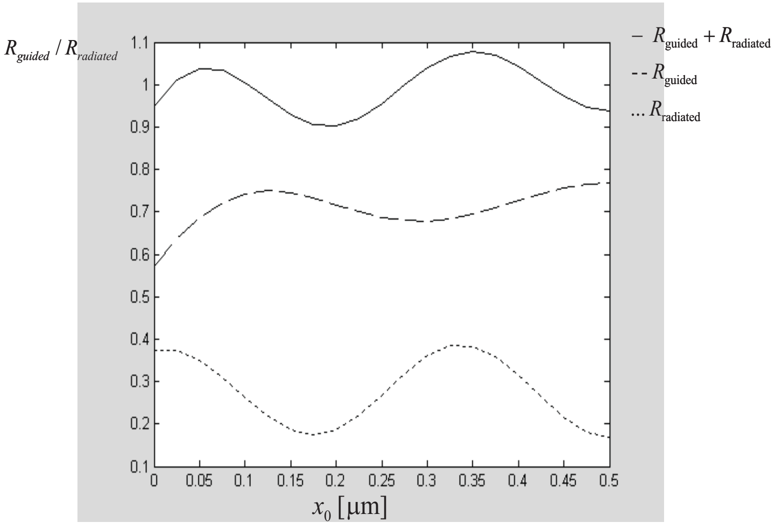

Figure 4.7: *case3:Three guided modes*

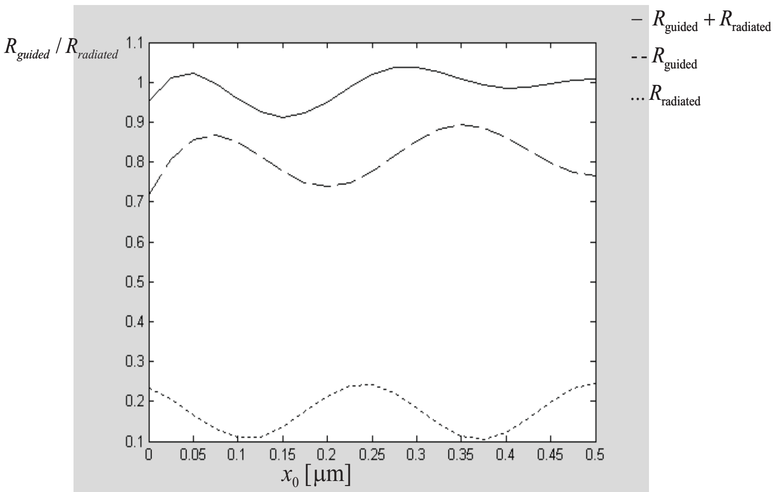

Figure 4.8: *case4: Four guided modes*

support the derived expressions.

4.4 Conclusion

A few examples and applications of the obtained expressions have been presented. It has been shown that the radiative power in a uniform space is equal to the guided power in a system closes with metal/magnetic walls in the limit of infinite separation as it should. The numerical examples (namely example1) showed that the relative power oscillates around 1 with an amplitude that gets smaller for large values of the thickness of the core layer as expected. These results give us confidence that the obtained expressions make sense.

Chapter 5

Summary and Conclusion

The power coming from a radiating dipole in a planar structure is considered. Analytical exact expressions for the radiated and for the guided power for both TE and TM polarization are obtained. In summary the following has been presented.

- **Chapter 1**
  In chapter 1 we stated the goal of this M.Sc thesis. We also gave an overview of the work that have been done so far on the same topic. Finally, the outline of the thesis is presented.

- **Chapter 2**
  In the very beginning of chapter 2 we defined the problem. Then we treated the twodimensional problem for both TE and TM polarization in the Fourier domain. We derived expressions for the amplitudes of the radiated and the guided Fields.

- **Chapter 3**
  In chapter 3 we extended the result of chapter 2 to a threedimensional formulation. To achieve this a rotation transform is used to reduce the spatial dimensionality for the threedimensional problem to twodimensional problem in rotated system. Then we derived expressions for the radiated and for the guided power.

- **Chapter 4**
  In chapter 4 we tested and investigated (theoretically) the validity of the obtained expressions for the radiated and the guided power. We

also gave numerical examples that have been calculated in Matlb environment.

From the obtained expressions and the numerical tests we can see that the transmitted power is strongly influence by the position of the radiating source. The tests have been done so far support the validity and the readability of the derived formulae. In general the obtained expression for the radiated power , though they are in integral form, and for the guided power they are simple, transparent and easy to evaluate (assuming that there is already a mode solver available).

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