A variational formulation of guided wave scattering problems

— hybrid analytic / numerical coupled mode modeling

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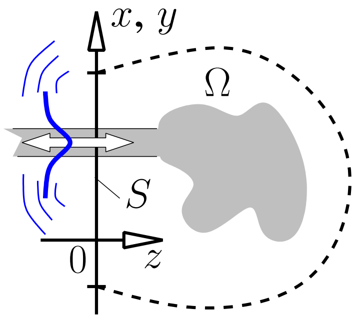
Abstract scattering problem



Ω: domain of interest,

$$\left.\begin{aligned} \nabla\times\boldsymbol{H} - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E} &= 0 \\ -\nabla\times\boldsymbol{E} - \mathrm{i}\omega\mu_0\boldsymbol{H} &= 0 \end{aligned}\right\} \text{ in } \Omega$$

for given frequency ω, permittivity $\epsilon = n^2$,

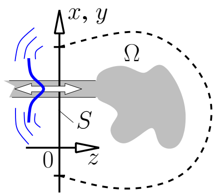
S: an exemplary port plane, waveguides enter Ω through S.

Variational form including suitable boundary conditions ?

Outline

- Guided wave scattering problems
  - Transparent influx boundary conditions
  - Variational formulation
- Hybrid analytic / numerical coupled mode theory
  - Template, discretization
  - Variational scheme
  - Results, waveguide crossing
- Comments

Boundary conditions



Ingredients:

- Complete set of normal modes on S, $(\tilde{\boldsymbol{E}}_m, \pm\tilde{\boldsymbol{H}}_m)(x, y) \longleftrightarrow$ propagation along $\pm z$.
- Product on S: $\langle \boldsymbol{A}, \boldsymbol{B}\rangle = \iint_S (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{e}_z \,\mathrm{d}x\,\mathrm{d}y$.
- Modal orthogonality properties $\langle \tilde{\boldsymbol{E}}_l, \tilde{\boldsymbol{H}}_k\rangle = \delta_{lk}N_k$, $N_k = \langle \tilde{\boldsymbol{E}}_k, \tilde{\boldsymbol{H}}_k\rangle$.

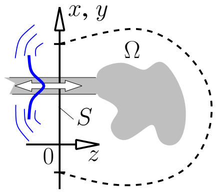
"Any" electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ on S can be expanded as

$$\boldsymbol{E} = \sum_m e_m\tilde{\boldsymbol{E}}_m, \quad e_m = \frac{1}{N_m}\langle \boldsymbol{E}, \tilde{\boldsymbol{H}}_m\rangle, \quad \boldsymbol{H} = \sum_m h_m\tilde{\boldsymbol{H}}_m, \quad h_m = \frac{1}{N_m}\langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H}\rangle,$$

or $$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} = \sum_m f_m \begin{pmatrix}\tilde{\boldsymbol{E}}_m\\ \tilde{\boldsymbol{H}}_m\end{pmatrix} + \sum_m b_m \begin{pmatrix}\tilde{\boldsymbol{E}}_m\\ -\tilde{\boldsymbol{H}}_m\end{pmatrix}, \qquad \begin{aligned} f_m &= (e_m + h_m)/2, \\ b_m &= (e_m - h_m)/2 \end{aligned}$$

(transverse components only).

Transparent influx boundary conditions (TIBCs)



... on S for inhomogeneous exterior, incoming waveguides:

$$\begin{aligned}\boldsymbol{E} &= \sum_m 2F_m\tilde{\boldsymbol{E}}_m - \sum_m \frac{1}{N_m}\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle\tilde{\boldsymbol{E}}_m\,,\\ \boldsymbol{H} &= \sum_m 2F_m\tilde{\boldsymbol{H}}_m - \sum_m \frac{1}{N_m}\langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle\tilde{\boldsymbol{H}}_m\,;\end{aligned}$$

F_m: influx, given coefficients of incoming waves; $\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}_{\text{inc}} = \sum_m F_m\begin{pmatrix}\tilde{\boldsymbol{E}}_m\\ \tilde{\boldsymbol{H}}_m\end{pmatrix}$.

For a general field of the form $\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} = \sum_m f_m\begin{pmatrix}\tilde{\boldsymbol{E}}_m\\ \tilde{\boldsymbol{H}}_m\end{pmatrix} + \sum_m b_m\begin{pmatrix}\tilde{\boldsymbol{E}}_m\\ -\tilde{\boldsymbol{H}}_m\end{pmatrix}$

the TIBCs require $f_m = F_m$, while b_m can be arbitrary.

Frequency domain Maxwell equations, variational form

Consider the functional

$$\mathcal{L}(\boldsymbol{E},\boldsymbol{H}) = \iiint_\Omega \{\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}) + \boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}^2 + \mathrm{i}\omega\mu_0\boldsymbol{H}^2\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

(C. Vassallo. *Optical Waveguide Concepts*. Elsevier, Amsterdam, 1991).

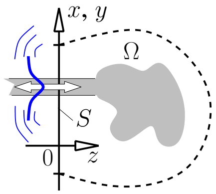
First variation:

$$\begin{aligned}\delta\mathcal{L}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) &= \iiint_\Omega \{2\delta\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H} - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E})\\ &\qquad + 2\delta\boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E} + \mathrm{i}\omega\mu_0\boldsymbol{H})\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\quad - \iint_{\partial\Omega}\{(\boldsymbol{n}\times\boldsymbol{E})\cdot\delta\boldsymbol{H} + (\boldsymbol{n}\times\boldsymbol{H})\cdot\delta\boldsymbol{E}\}\,\mathrm{d}A\,.\end{aligned}$$

Stationarity $\delta\mathcal{L}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) = 0$ for arbitrary $\delta\boldsymbol{E}$, $\delta\boldsymbol{H}$ implies

- that $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy the Maxwell equations in Ω
- and that transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ vanish on $\partial\Omega$.

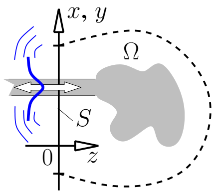
Variational form of the scattering problem



... based on the functional:

$$\begin{aligned}\mathcal{F}(\boldsymbol{E},\boldsymbol{H}) = &\iiint_\Omega \{\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}) + \boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}^2 + \mathrm{i}\omega\mu_0\boldsymbol{H}^2\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &- \sum_m 2F_m\left\{\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle - \langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle\right\}\\ &+ \sum_m \frac{1}{2N_m}\left\{\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle^2 - \langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle^2\right\}\end{aligned}$$

Variational form of the scattering problem, first variation

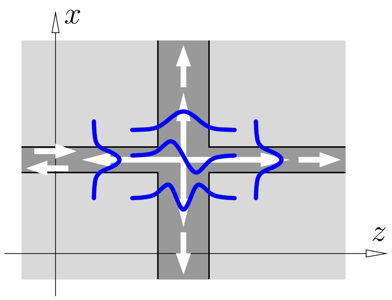


$$\begin{aligned}\delta\mathcal{F}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) &= \iiint_\Omega \{2\delta\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H} - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E})\\ &\qquad + 2\delta\boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E} + \mathrm{i}\omega\mu_0\boldsymbol{H})\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\quad + \Big\langle\boldsymbol{E} - \sum_m 2F_m\tilde{\boldsymbol{E}}_m + \sum_m\frac{1}{N_m}\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle\tilde{\boldsymbol{E}}_m,\,\delta\boldsymbol{H}\Big\rangle\\ &\quad - \Big\langle\delta\boldsymbol{E},\,\boldsymbol{H} - \sum_m 2F_m\tilde{\boldsymbol{H}}_m + \sum_m\frac{1}{N_m}\langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle\tilde{\boldsymbol{H}}_m\Big\rangle\\ &\quad - \iint_{\partial\Omega\backslash S}\{(\boldsymbol{n}\times\boldsymbol{E})\cdot\delta\boldsymbol{H} + (\boldsymbol{n}\times\boldsymbol{H})\cdot\delta\boldsymbol{E}\}\,\mathrm{d}A\,.\end{aligned}$$

Stationarity $\delta\mathcal{F}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) = 0$ for arbitrary $\delta\boldsymbol{E}$, $\delta\boldsymbol{H}$ implies

- that $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy the Maxwell equations in Ω,
- that $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy TIBCs on S,
- and that transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ vanish on $\partial\Omega\backslash S$.

Hybrid coupled mode theory: waveguide crossing



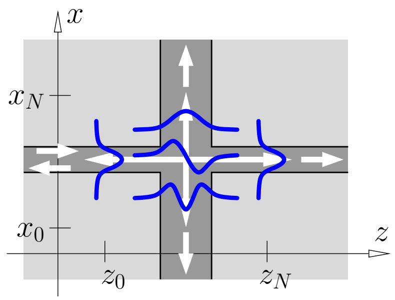
Basis elements:

- guided modes of the horizontal WG
  $$\boldsymbol{\psi}_m^{\mathrm{f,b}}(x,z) = \begin{pmatrix}\tilde{\boldsymbol{E}}\\ \tilde{\boldsymbol{H}}\end{pmatrix}_m^{\mathrm{f,b}}(x)\,\mathrm{e}^{\mp\mathrm{i}\beta_m^{\mathrm{f,b}} z},$$
- guided modes of the vertical WG
  $$\boldsymbol{\psi}_m^{\mathrm{u,d}}(x,z) = \begin{pmatrix}\tilde{\boldsymbol{E}}\\ \tilde{\boldsymbol{H}}\end{pmatrix}_m^{\mathrm{u,d}}(z)\,\mathrm{e}^{\mp\mathrm{i}\beta_m^{\mathrm{u,d}} x}$$

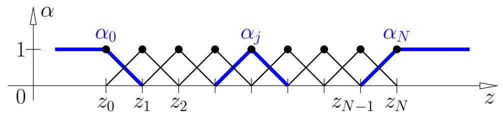
$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,z) = \sum_m f_m(z)\boldsymbol{\psi}_m^{\mathrm{f}}(x,z) + \sum_m b_m(z)\boldsymbol{\psi}_m^{\mathrm{b}}(x,z) + \sum_m u_m(x)\boldsymbol{\psi}_m^{\mathrm{u}}(x,z) + \sum_m d_m(x)\boldsymbol{\psi}_m^{\mathrm{d}}(x,z)$$

$f_m,\, b_m,\, u_m,\, d_m$: ?

Amplitude functions, discretization



1-D linear finite elements



$$f_m(z) = \sum_{j=0}^{N} f_{m,j}\,\alpha_j(z)\,,$$

$b_m(z)$, $u_m(x)$, $d_m(x)$ analogous.

$$\hookrightarrow \quad \begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,z) = \sum_k a_k \Big(\alpha_{\cdot}(\cdot)\boldsymbol{\psi}_{\cdot}^{\cdot}(x,z)\Big) =: \sum_k a_k \begin{pmatrix}\boldsymbol{E}_k\\ \boldsymbol{H}_k\end{pmatrix}(x,z),$$

$k \in \{\text{waveguides, modes, elements}\}$, $\quad a_k \in \{f_{m,j}, b_{m,j}, u_{m,j}, d_{m,j}\}$.

Variational HCMT scheme

$$\mathcal{F}(\boldsymbol{E},\boldsymbol{H}) \xrightarrow{(\boldsymbol{E},\boldsymbol{H}) = \sum_k a_k(\boldsymbol{E}_k,\boldsymbol{H}_k)} \mathcal{F}_{\mathrm{r}}(\boldsymbol{a})$$

Restricted functional:

$$\mathcal{F}_{\mathrm{r}}(\boldsymbol{a}) = \sum_{l,k} a_l F_{lk} a_k + \sum_l R_l a_l + \sum_{l,k} a_l B_{lk} a_k\,,$$

$$F_{lk} = \iiint_\Omega \{\boldsymbol{E}_l \cdot (\boldsymbol{\nabla}\times\boldsymbol{H}_k) + \boldsymbol{H}_l\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}_k) - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}_l\cdot\boldsymbol{E}_k + \mathrm{i}\omega\mu_0\boldsymbol{H}_l\cdot\boldsymbol{H}_k\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,,$$

$$R_l = -\sum_m 2F_m\left\{\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}_l\rangle - \langle\boldsymbol{E}_l,\tilde{\boldsymbol{H}}_m\rangle\right\},$$

$$B_{lk} = \sum_m \frac{1}{2N_m}\left\{\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}_l\rangle\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}_k\rangle - \langle\boldsymbol{E}_l,\tilde{\boldsymbol{H}}_m\rangle\langle\boldsymbol{E}_k,\tilde{\boldsymbol{H}}_m\rangle\right\},$$

+ contributions R, B from other port planes.

Variational HCMT scheme

$$\mathcal{F}(\boldsymbol{E},\boldsymbol{H}) \xrightarrow{(\boldsymbol{E},\boldsymbol{H}) = \sum_k a_k(\boldsymbol{E}_k,\boldsymbol{H}_k)} \mathcal{F}_{\mathrm{r}}(\boldsymbol{a})$$

Restricted functional:

$\mathcal{F}_{\mathrm{r}}(\boldsymbol{a}) = \boldsymbol{a}\cdot\mathsf{M}\boldsymbol{a} + \boldsymbol{R}\cdot\boldsymbol{a}.$

Require $\delta\mathcal{F}_{\mathrm{r}} = \delta\boldsymbol{a}\cdot\Big(\big(\mathsf{M}+\mathsf{M}^{\mathsf{T}}\big)\,\boldsymbol{a} + \boldsymbol{R}\Big) = 0$ for all $\delta\boldsymbol{a}$,

$\hookrightarrow$

$\big(\mathsf{M}+\mathsf{M}^{\mathsf{T}}\big)\,\boldsymbol{a} + \boldsymbol{R} = 0,$

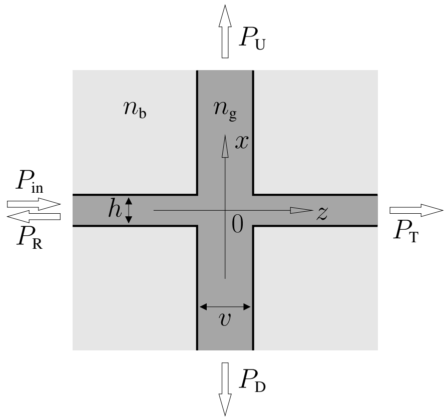
$\hookrightarrow$

$\boldsymbol{a},$ (...)

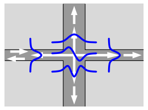
$\hookrightarrow$

$f_m,\ b_m,\ u_m,\ d_m,\ \ \boldsymbol{E},\ \boldsymbol{H}.$

Waveguide crossing, results



$n_g = 3.4$, $n_b = 1.45$, $\lambda = 1.55\,\mu$m,
$h = 0.2\,\mu$m, v variable, TE polarization.



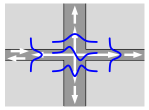
Basis elements:
guided modes of the horizontal and vertical cores (directional variants).

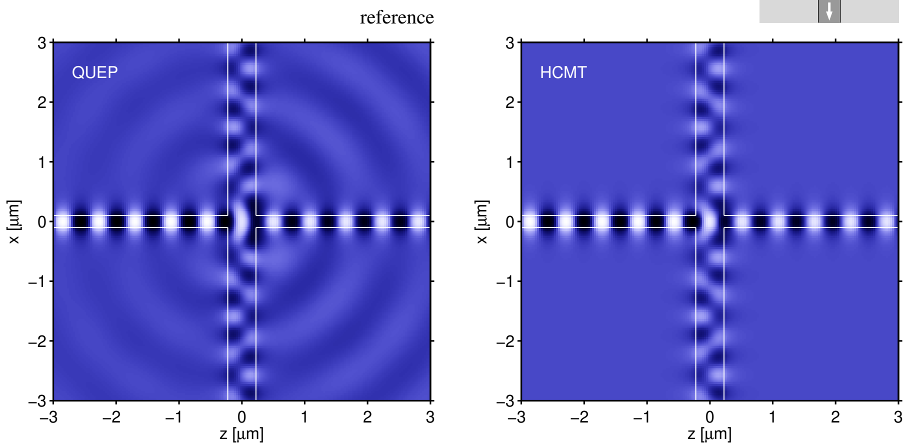
FEM discretization:
$z \in [v/2 - 1.5\,\mu\text{m}, v/2 + 1.5\,\mu\text{m}]$, $\Delta x = 0.025\,\mu$m,
$x \in [w/2 - 1.5\,\mu\text{m}, w/2 + 1.5\,\mu\text{m}]$, $\Delta z = 0.025\,\mu$m.

Computational window:
$z \in [-4\,\mu\text{m}, 4\,\mu\text{m}]$, $x \in [-4\,\mu\text{m}, 4\,\mu\text{m}]$.

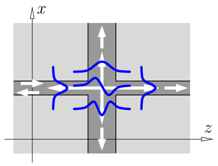
Waveguide crossing, fields

$v = 0.45\,\mu$m:

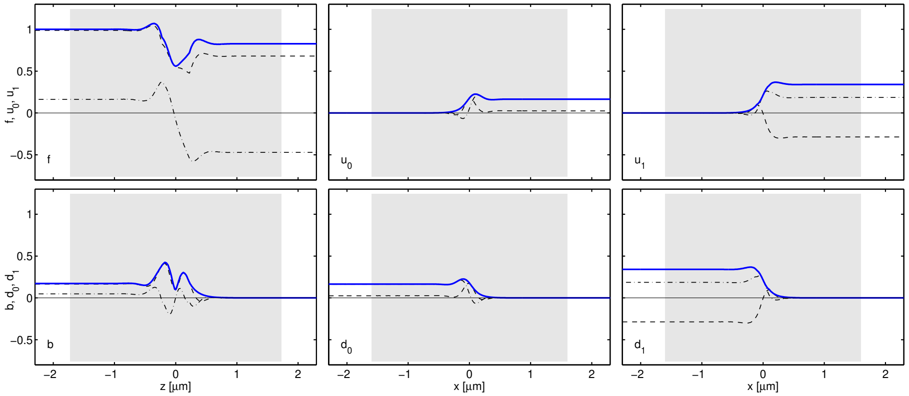




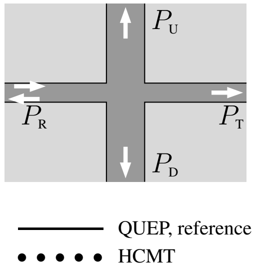
Waveguide crossing, amplitude functions

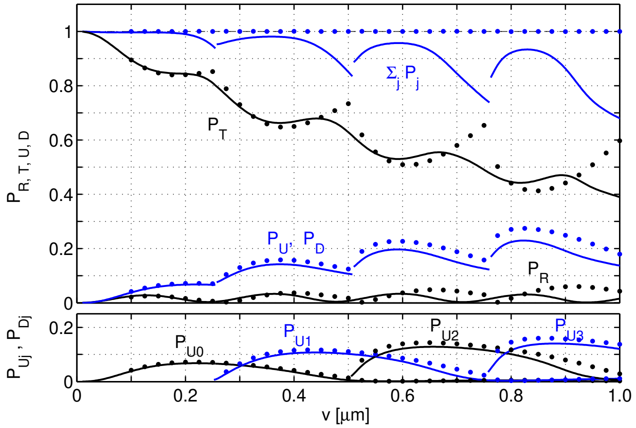


$v = 0.45\,\mu$m:



Waveguide crossing, power transfer

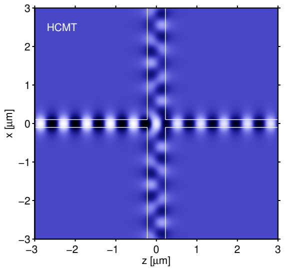




Comments

HCMT scheme based on a variational form of the guided wave scattering problem:

- Expansions at the TIBC ports reduce to single terms due to modal orthogonality.
- Bidirectional basis fields are required for all channels in the field templates.



Alternative functional:

$$\mathcal{C}(\boldsymbol{E},\boldsymbol{H}) = \iiint_\Omega \big\{ \boldsymbol{E}^{*}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}) - \boldsymbol{H}^{*}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) \\ - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}^{*}\cdot\boldsymbol{E} + \mathrm{i}\omega\mu_0\boldsymbol{H}^{*}\cdot\boldsymbol{H}\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Extend $\mathcal{C}$ by boundary integrals such that

- the boundary terms in $\delta\mathcal{C}$ cancel $\leftrightsquigarrow$ the Galerkin scheme (OWTNM'06) could be viewed as a variational restriction of $\mathcal{C}$.
- TIBCs are satisfied as natural boundary conditions if $\mathcal{C}$ becomes stationary $\leftrightsquigarrow$ variational scheme with complex conjugate fields.