# Geometry Tolerance Estimation for Rectangular Dielectric Waveguide Devices by Means of Perturbation Theory

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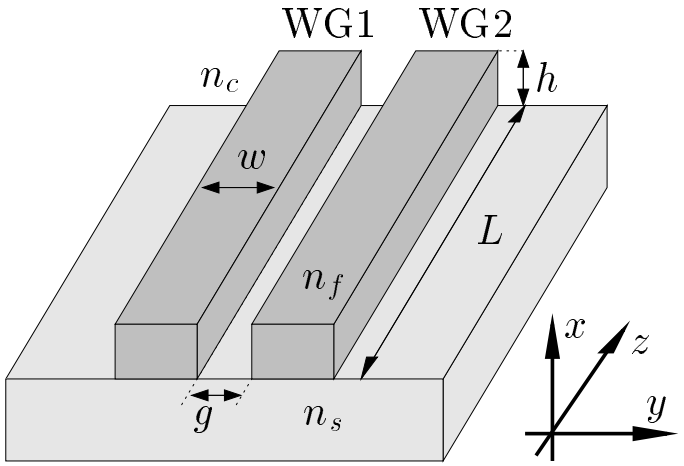
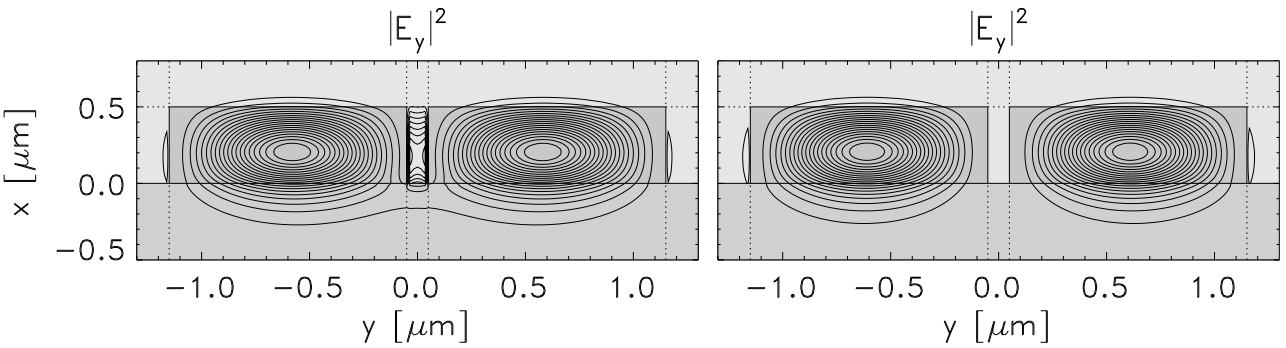



Guided modes, propagation constants $\beta_\mathsf{s}$, $\beta_\mathsf{a}$:



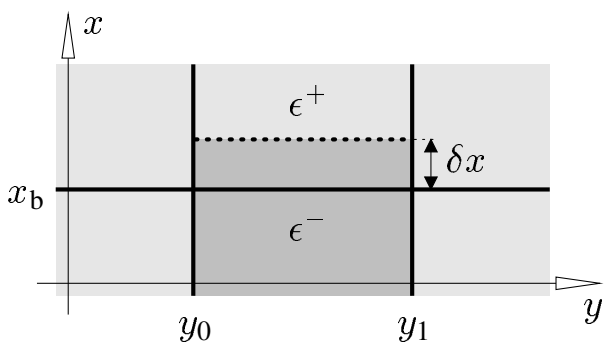

Power transfer WG1 $\longrightarrow$ WG2: $\quad P_{1\to 2} \approx \dfrac{1}{2}(1-\cos(\beta_\mathsf{s}-\beta_\mathsf{a})L)$ .

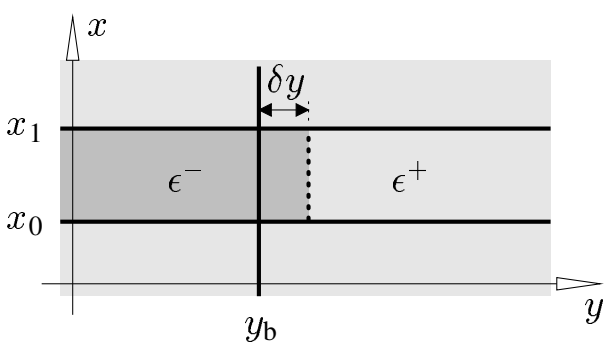
$q$: any geometry parameter.

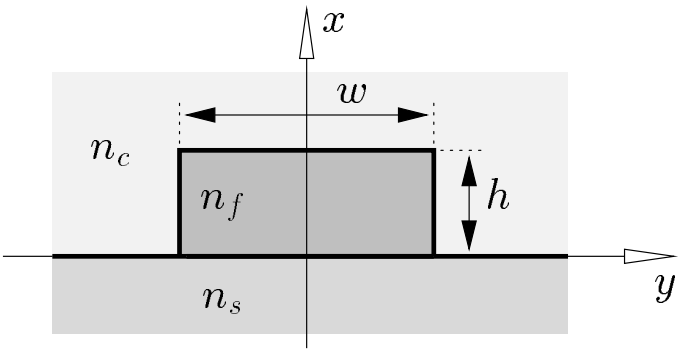
$\partial_q P$ ?

$\partial_q \beta$ ?

Shift of a boundary in the dielectric profile
$\leftrightarrow$ thin layer perturbations on the strip surface/sidewalls . . .






Guided modes: electromagnetic fields of the form

$$\boldsymbol{E}(x,y)\ \mathrm{e}^{\mathrm{i}(\omega t-\beta z)},\quad \boldsymbol{H}(x,y)\ \mathrm{e}^{\mathrm{i}(\omega t-\beta z)},$$

$\omega\sqrt{\epsilon_0\mu_0}=k=2\pi/\lambda$ for wavelength $\lambda$.
$\boldsymbol{E}(x,y)$, $\boldsymbol{H}(x,y)$: the square integrable mode profile,
$\beta$: the corresponding propagation constant.

Maxwell's curl equations:

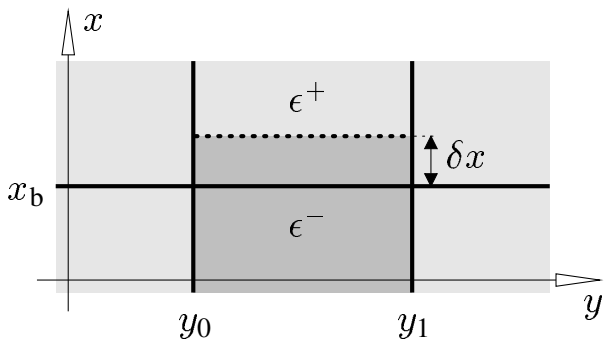
$$(\mathsf{C}+\mathrm{i}\beta\mathsf{R})\boldsymbol{E}=-\mathrm{i}\omega\mu_0\boldsymbol{H},\quad (\mathsf{C}+\mathrm{i}\beta\mathsf{R})\boldsymbol{H}=\mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E},$$

$$\mathsf{C}=\begin{pmatrix}0&0&\partial_y\\0&0&-\partial_x\\-\partial_y&\partial_x&0\end{pmatrix},\quad \mathsf{R}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}.$$

$(\boldsymbol{F},\boldsymbol{G}):=\iint \boldsymbol{F}^*\boldsymbol{G}\,\mathrm{d}x\,\mathrm{d}y$;
Define $\mathsf{B}_\epsilon$ as a functional of the independent fields $E_x,\ldots,H_z$:

$$\mathsf{B}_\epsilon(\boldsymbol{E},\boldsymbol{H})=\frac{\omega\epsilon_0(\boldsymbol{E},\epsilon\boldsymbol{E})+\omega\mu_0(\boldsymbol{H},\boldsymbol{H})+\mathrm{i}(\boldsymbol{E},\mathsf{C}\boldsymbol{H})-\mathrm{i}(\boldsymbol{H},\mathsf{C}\boldsymbol{E})}{(\boldsymbol{E},\mathsf{R}\,\boldsymbol{H})-(\boldsymbol{H},\mathsf{R}\,\boldsymbol{E})}.$$

$\mathsf{B}_\epsilon$ is stationary at a valid mode field with $\beta=\mathsf{B}_\epsilon(\boldsymbol{E},\boldsymbol{H})$.

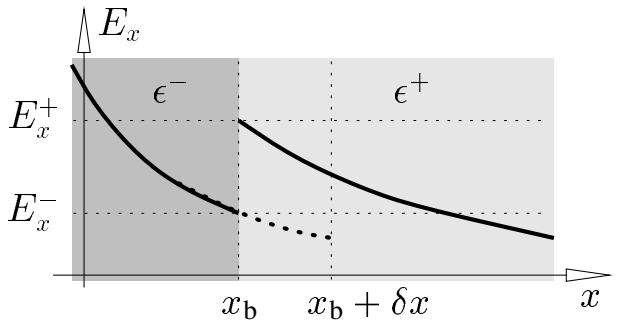



□: the region
$x_\mathrm{b} < x < x_\mathrm{b} + \delta x$,
$y_0 < y < y_1$.

Shifting the boundary from $x_\mathrm{b}$ to $x_\mathrm{b} + \delta x$ changes
$\epsilon \longrightarrow \epsilon + \delta\epsilon$,
$\beta \longrightarrow \beta + \delta\beta$,
$(\boldsymbol{E}, \boldsymbol{H}) \longrightarrow (\boldsymbol{E} + \delta\boldsymbol{E}, \boldsymbol{H} + \delta\boldsymbol{H})$,

$$\delta\epsilon\,(x,y) = \begin{cases} (\epsilon^- - \epsilon^+) & \text{for } (x,y) \in \square, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta + \delta\beta = \mathsf{B}_{\epsilon+\delta\epsilon}(\boldsymbol{E} + \delta\boldsymbol{E}, \boldsymbol{H} + \delta\boldsymbol{H}).$$

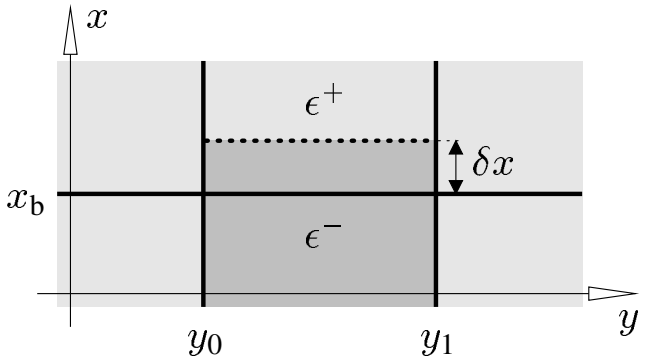



$\delta\boldsymbol{E},\ \delta\boldsymbol{H}$ ?

- $E_y$, $E_z$, $H_x$, $H_y$, $H_z$ are continuous in $x_\mathrm{b}$ and in $x_\mathrm{b} + \delta x$
  $\rightsquigarrow \delta E_y = \delta E_z = 0$, $\delta H_x = \delta H_y = \delta H_z = 0$.
- $E_x$ jumps in $x_\mathrm{b}$: $\epsilon^- E_x^- = \epsilon^+ E_x^+$, but is continuous in $x_\mathrm{b} + \delta x$
  $\rightsquigarrow$
  $$\delta E_x(x,y) = \begin{cases} \dfrac{(\epsilon^+ - \epsilon^-)}{\epsilon^-} E_x(x,y) & \text{for } (x,y) \in \square, \\ 0 & \text{otherwise.} \end{cases}$$
  $E_x + \delta E_x$ is continuous in $x_\mathrm{b}$, but jumps in $x_\mathrm{b} + \delta x$.

**Result** (C. Vassallo: *Optical Waveguide Concepts*, Elsevier 1991):

$$\delta\beta = \frac{\omega\epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \iint_\square \left( \dfrac{\epsilon^+}{\epsilon^-} |E_x|^2 + |E_y|^2 + |E_z|^2 \right) \mathrm{d}x\,\mathrm{d}y}{\iint \left( E_x H_y^* - E_y H_x^* \right) \mathrm{d}x\,\mathrm{d}y}.$$




*Horizontal* boundary shifts:

- hybrid fields

$$\delta\beta = \frac{\omega\epsilon_0}{2}\,\frac{(\epsilon^- - \epsilon^+)\displaystyle\int_{y_0}^{y_1}\left((E_x^+)^* E_x^- + |E_y|^2 + |E_z|^2\right)(x_\mathrm{b}, y)\,\mathrm{d}y}{\displaystyle\iint\left(E_x H_y^* - E_y H_x^*\right)\,\mathrm{d}x\,\mathrm{d}y}\,\delta x\,,$$

- TE-polarized semivectorial fields
  $E_x := 0$, $\beta^{-1}\partial_y E_y \approx 0$ assumed for $x = x_\mathrm{b}$, $y_0 < y < y_1$,

$$\delta\beta = \frac{k_0^2}{2\beta}\,\frac{(\epsilon^- - \epsilon^+)\displaystyle\int_{y_0}^{y_1}|E_y|^2(x_\mathrm{b}, y)\,\mathrm{d}y}{\displaystyle\iint|E_y|^2\mathrm{d}x\,\mathrm{d}y}\,\delta x\,,$$

- TM-polarized semivectorial fields
  $H_x := 0$, $\beta^{-1}\partial_y H_y \approx 0$ assumed for $x = x_\mathrm{b}$, $y_0 < y < y_1$,

$$\delta\beta = \frac{\beta}{2}\,\frac{(\epsilon^- - \epsilon^+)\displaystyle\int_{y_0}^{y_1}\left(\frac{1}{\epsilon^-\epsilon^+}|H_y|^2 + \beta^{-2}|\frac{1}{\epsilon}\partial_x H_y|^2\right)(x_\mathrm{b}, y)\,\mathrm{d}y}{\displaystyle\iint\frac{1}{\epsilon}|H_y|^2\,\mathrm{d}x\,\mathrm{d}y}\,\delta x\,.$$

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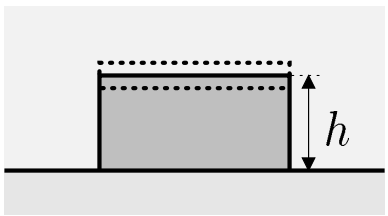
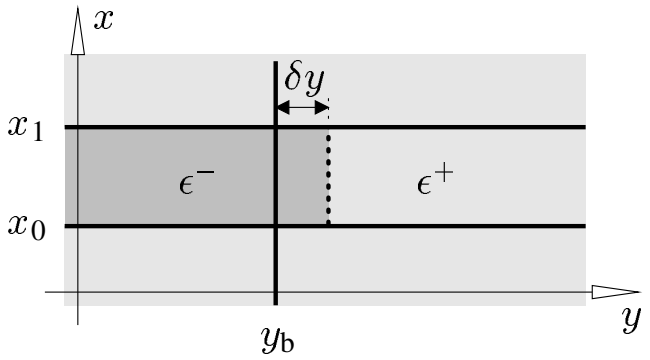



$\rightsquigarrow \quad \partial_h\beta.$




*Vertical* boundary shifts:

- hybrid fields

$$\delta\beta = \frac{\omega\epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \int_{x_0}^{x_1} \left( |E_x|^2 + (E_y^+)^* E_y^- + |E_z|^2 \right)(x, y_\mathsf{b})\, \mathsf{d}x}{\iint \left( E_x H_y^* - E_y H_x^* \right) \mathsf{d}x\, \mathsf{d}y} \delta y \,,$$

- TE-polarized semivectorial fields
  $E_x := 0$

$$\delta\beta = \frac{k_0^2}{2\beta} \frac{(\epsilon^- - \epsilon^+) \int_{x_0}^{x_1} \left( (E_y^+)^* E_y^- + \beta^{-2} |\partial_y E_y|^2 \right)(x, y_\mathsf{b})\, \mathsf{d}x}{\iint \left( |E_y|^2 - \beta^{-2} E_y \partial_y^2 E_y^* \right) \mathsf{d}x\, \mathsf{d}y} \delta y \,,$$

- TM-polarized semivectorial fields
  $H_x := 0, \quad (?)$

$$\delta\beta = \frac{k_0^2}{2\beta} \frac{(\epsilon^- - \epsilon^+) \int_{x_0}^{x_1} |H_y|^2 (x, y_\mathsf{b})\, \mathsf{d}x}{\iint |H_y|^2 \mathsf{d}x\, \mathsf{d}y} \delta y \,.$$

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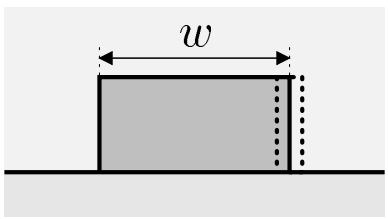
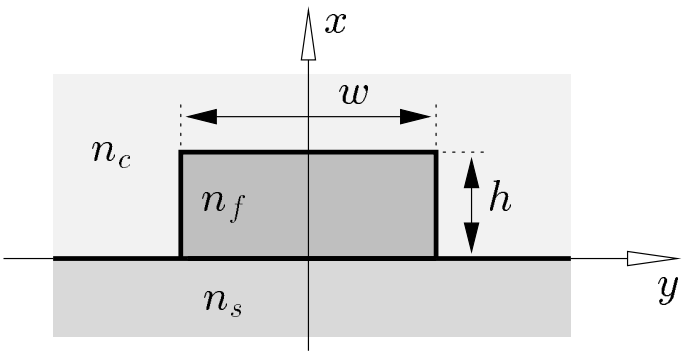



$\rightsquigarrow \quad \partial_w \beta.$




- Refractive indices:
  Use common formulas for small uniform bulk perturbations
  $\rightsquigarrow \quad \partial_{n_\mathrm{s}}\beta$, $\partial_{n_\mathrm{f}}\beta$, $\partial_{n_\mathrm{c}}\beta$.

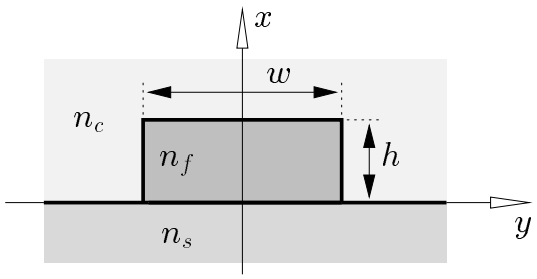
- Wavelength:

$$\beta(x\lambda, xw, xh) = \frac{1}{x}\beta(\lambda, w, h)$$

$$\rightsquigarrow \quad \partial_\lambda\beta = -\frac{1}{\lambda}\left(\beta + w\partial_w\beta + h\partial_h\beta\right).$$

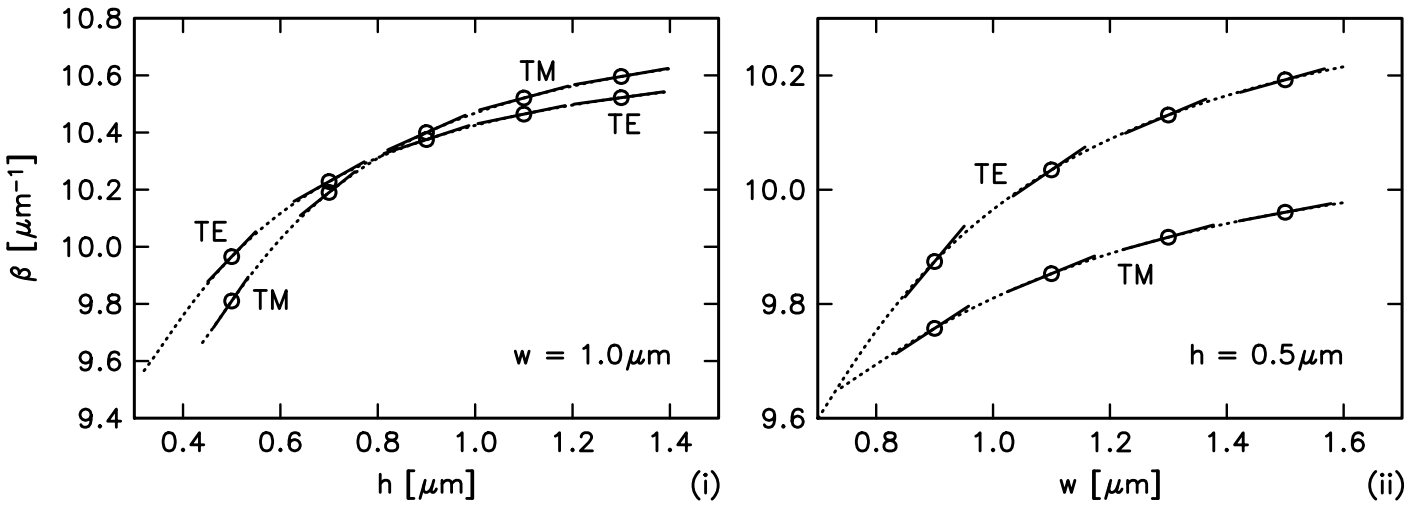
---

$\partial_h\beta$, $\partial_w\beta$, $\partial_{n_\mathrm{s}}\beta$, $\partial_{n_\mathrm{f}}\beta$, $\partial_{n_\mathrm{c}}\beta$, and $\partial_\lambda\beta$ are at hand.




$n_s = 1.95$, $n_f = 2.3$, $n_c = 1.0$, $\lambda = 1.3\,\mu m$

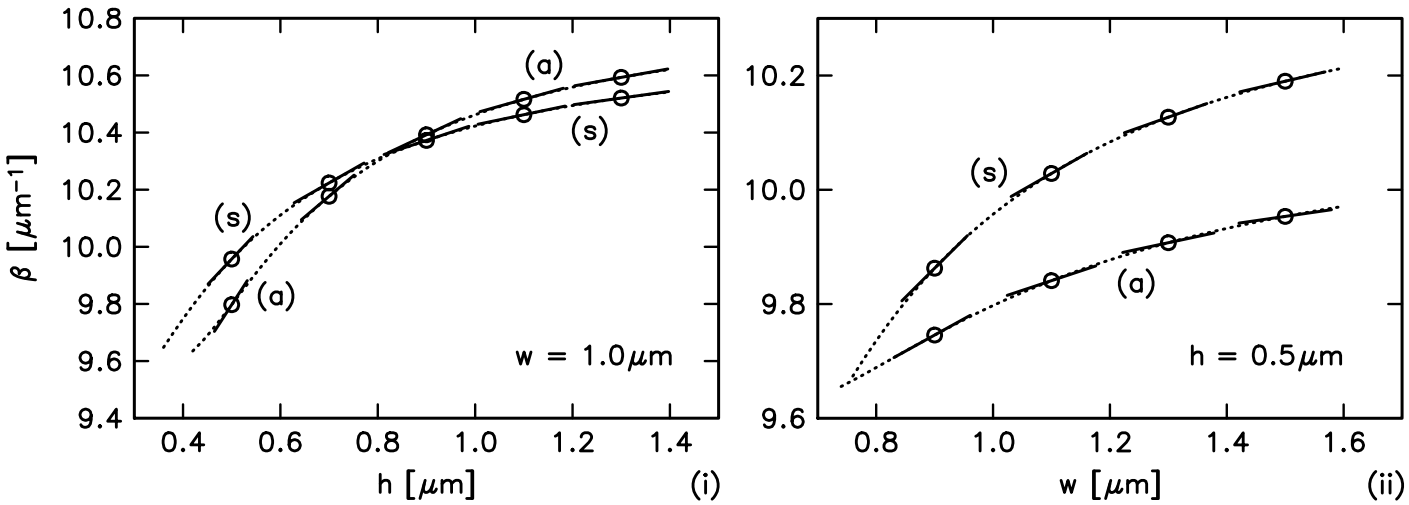
- Semivectorial fields, fundamental modes:



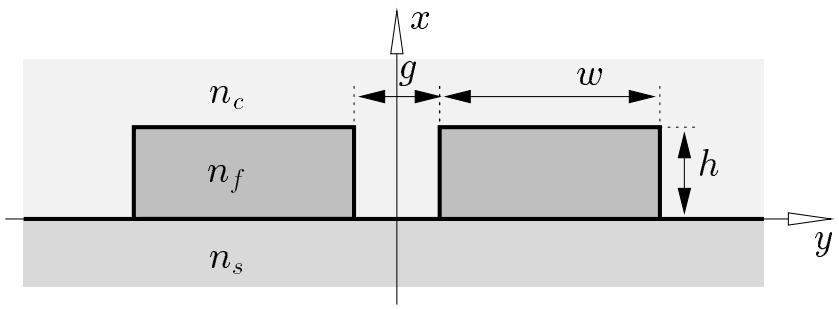

····· : direct computation $\beta(h)$, $\beta(w)$

—○— : $\beta(h_\circ) + (h - h_\circ)\,\partial_h\beta|_{h_\circ}$, $\beta(w_\circ) + (w - w_\circ)\,\partial_w\beta|_{w_\circ}$
with $\partial_h\beta$, $\partial_w\beta$ evaluated by the perturbation formulas

- Vectorial fields, fundamental modes of definite symmetry:

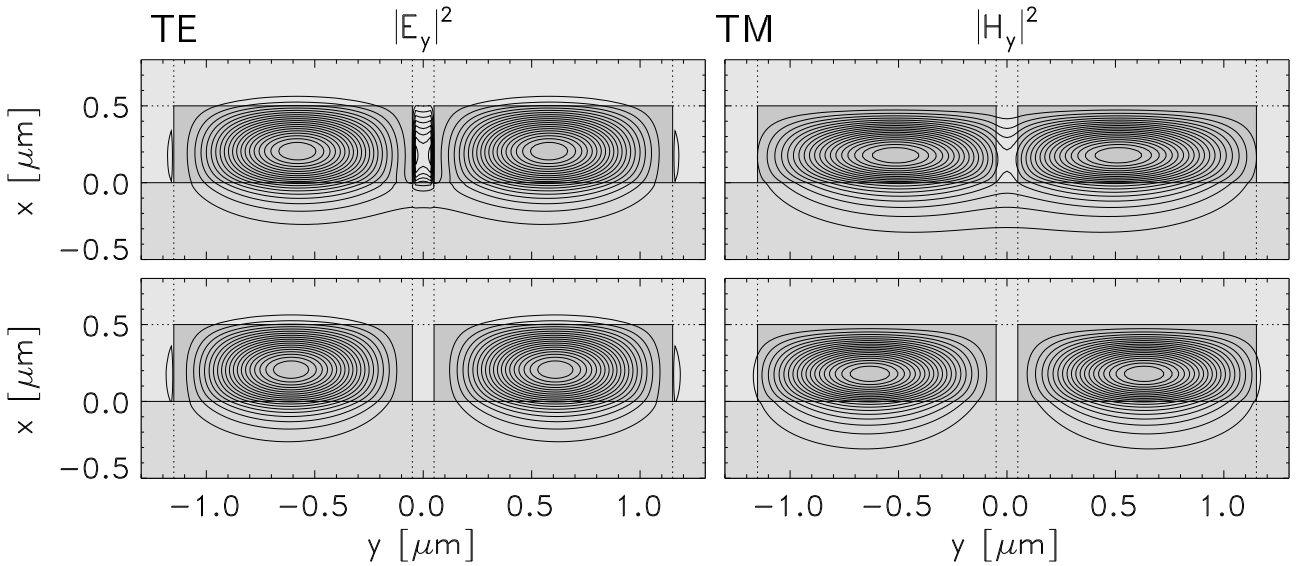



| | Symmetry $y \to -y$ | | Polarization-character |
|---|---|---|---|
| | $H_x$, $E_y$, $H_z$ | $E_x$, $H_y$, $E_z$ | |
| (s) | even | odd | TE |
| (a) | odd | even | TM |




$n_\mathsf{s} = 1.95$, $n_\mathsf{f} = 2.3$,
$n_\mathsf{c} = 1.0$, $\lambda = 1.3\,\mu\mathsf{m}$,
$h = 0.5\,\mu\mathsf{m}$, $w = 1.1\,\mu\mathsf{m}$,
$g = 0.1\,\mu\mathsf{m}$

Mode profiles:




Power transmission:

$\phi_\mathsf{l}$, $\phi_\mathsf{r}$: the modes of the isolated left and right channels
$\psi_\mathsf{s}$, $\psi_\mathsf{a}$: the symmetric/antisymmetric supermode
$\beta_\mathsf{s}$, $\beta_\mathsf{a}$: their propagation constants
$\langle\, .\,,\, .\,\rangle$: a suitable bilinear product

$\rightsquigarrow$ power output to $\phi_\mathsf{r}$ at $z = L$, excitation by $\phi_\mathsf{l}$ in $z = 0$:

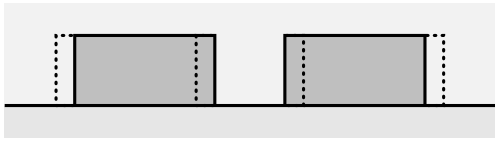
$$P(L) = w_\mathsf{s}^2 + w_\mathsf{a}^2 + 2w_\mathsf{s}w_\mathsf{a}\cos(\beta_\mathsf{s} - \beta_\mathsf{a})L$$
$$w_j = \langle\phi_\mathsf{r}, \psi_j\rangle\,\langle\psi_j, \phi_\mathsf{l}\rangle,\quad j = \mathsf{s}, \mathsf{a},\quad w_\mathsf{s}w_\mathsf{a} < 0.$$

Coupling length: $L_\mathsf{c} = \pi/(\beta_\mathsf{s} - \beta_\mathsf{a})$

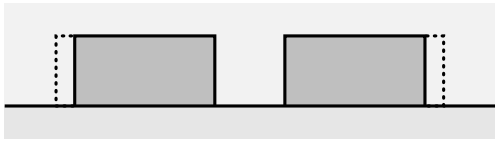
| | $\beta_\mathsf{s}/\,\mu\mathsf{m}^{-1}$ | $\beta_\mathsf{a}/\,\mu\mathsf{m}^{-1}$ | $L_\mathsf{c}/\,\mu\mathsf{m}$ |
|---|---|---|---|
| TE | 10.0505 | 10.0121 | 81.8 |
| TM | 9.9021 | 9.8095 | 33.9 |

Influence of geometry alterations $q \longrightarrow q + \delta q$ on the power transmission, $q \in \{g, w, h\}$:

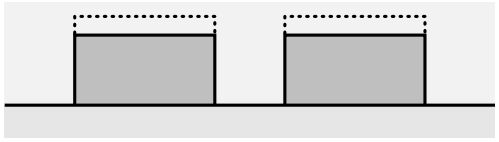
$q = g$:

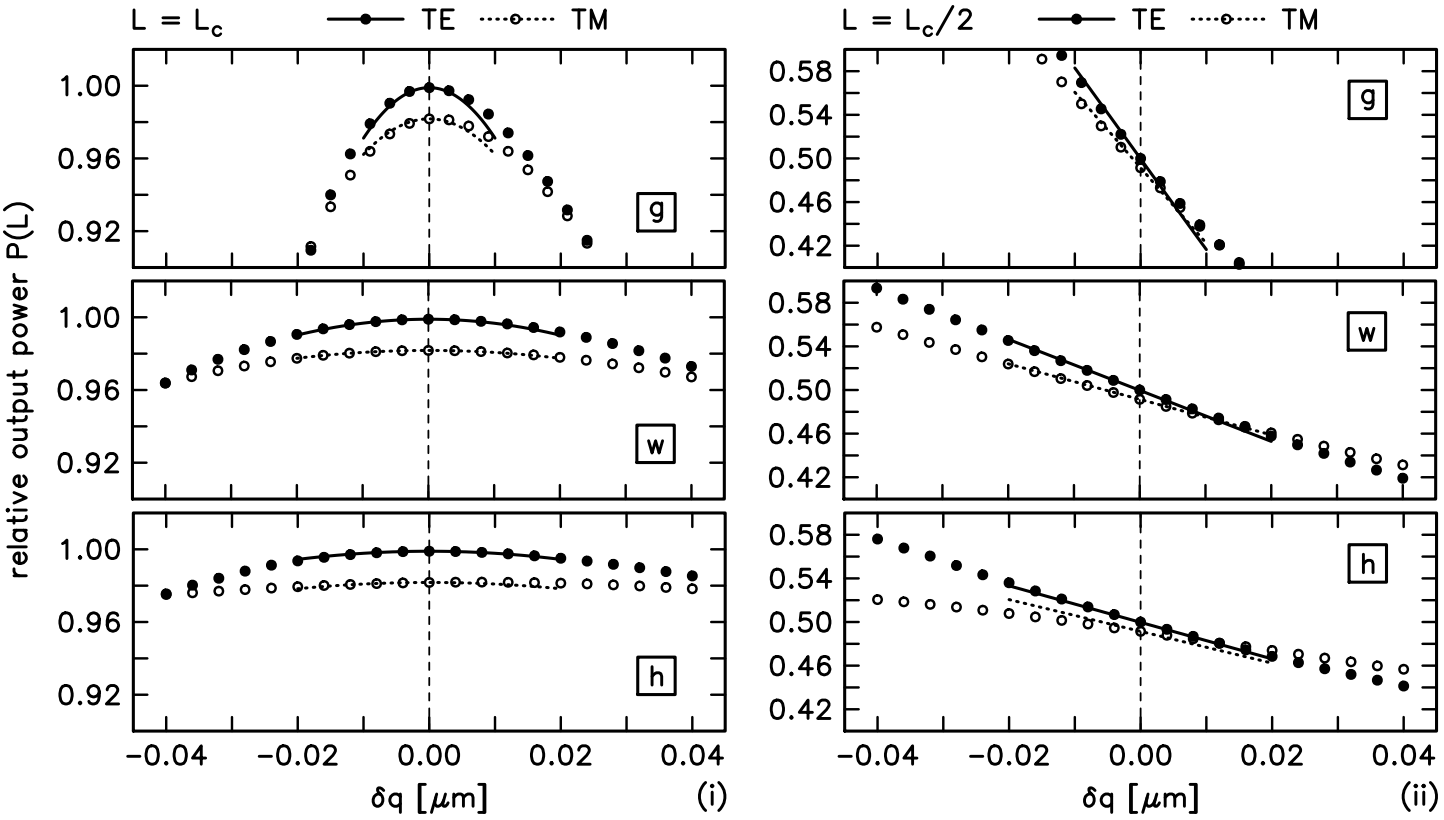


$q = w$:



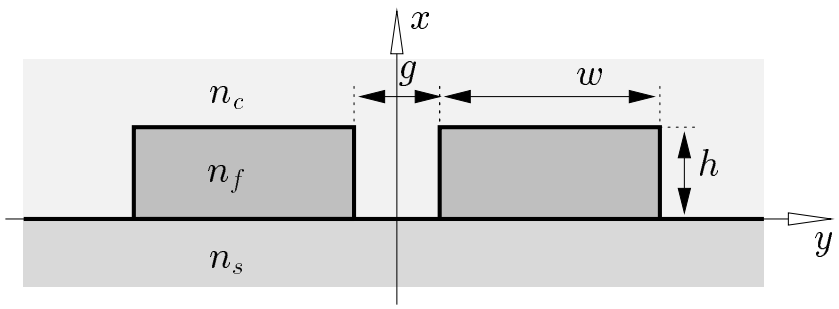
$q = h$:






Markers: $P(L)$ evaluated with directly calculated $\beta(q + \delta q)$
Lines: $P(L)$ evaluated with perturbational estimates $\beta(q) + \delta q\, \partial_q \beta$

$$P(L) = w_{\mathrm{s}}^2 + w_{\mathrm{a}}^2 + 2 w_{\mathrm{s}} w_{\mathrm{a}} \cos(\beta_{\mathrm{s}} - \beta_{\mathrm{a}}) L$$




$n_\mathsf{s} = 1.95$, $n_\mathsf{f} = 2.3$,
$n_\mathsf{c} = 1.0$, $\lambda = 1.3\,\mu\mathsf{m}$,
$h = 0.5\,\mu\mathsf{m}$, $w = 1.1\,\mu\mathsf{m}$,
$g = 0.1\,\mu\mathsf{m}$

Tolerances:

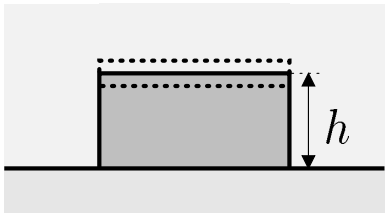
Assume $|w_\mathsf{s}| \approx |w_\mathsf{a}| \approx 1/2$; $P_q(L)$ depends on a parameter $q$ mainly via the propagation constants.

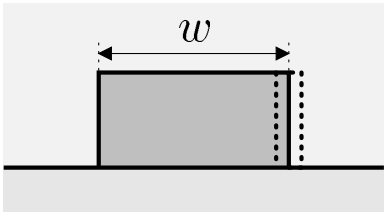
$\rightsquigarrow$ Expanding $P_{q+\delta q}(L)$ yields tolerance formulas for $q \in \{g, w, h, \lambda, n_\mathsf{s}, n_\mathsf{f}, n_\mathsf{c}\}$:

- $L = L_\mathsf{c}$:
$$\Delta q = \frac{2\sqrt{\Delta P}}{L|\partial_q\beta_\mathsf{s} - \partial_q\beta_\mathsf{a}|},$$
$$P_{q+\delta q} > P_q - \Delta P \quad \text{for} \quad -\Delta q < \delta q < \Delta q.$$

- $L = L_\mathsf{c}/2$:
$$\Delta q = \frac{2\Delta P}{L|\partial_q\beta_\mathsf{s} - \partial_q\beta_\mathsf{a}|},$$
$$P_q - \Delta P < P_{q+\delta q} < P_q + \Delta P \quad \text{for} \quad -\Delta q < \delta q < \Delta q.$$

| | | TE | TM | TE | TM | TE | TM |
|---|---|---|---|---|---|---|---|
| | | $\partial_g\beta/\mu\mathsf{m}^{-2}$ | | $\partial_w\beta/\mu\mathsf{m}^{-2}$ | | $\partial_h\beta/\mu\mathsf{m}^{-2}$ | |
| (s) | FD | $-0.280$ | $-0.565$ | $0.538$ | $0.284$ | $1.738$ | $2.427$ |
| | PT | $-0.335$ | $-0.636$ | $0.622$ | $0.293$ | $1.768$ | $2.455$ |
| (a) | FD | $0.075$ | $0.201$ | $0.645$ | $0.475$ | $1.820$ | $2.602$ |
| | PT | $0.075$ | $0.199$ | $0.737$ | $0.489$ | $1.850$ | $2.559$ |
| $\Delta P = 1\%$ | | $\Delta g/\mathsf{nm}$ | | $\Delta w/\mathsf{nm}$ | | $\Delta h/\mathsf{nm}$ | |
| $L = L_\mathsf{c}$ | NUM | $6.4$ | $6.6$ | $22$ | $30$ | $27$ | $50$ |
| | PT | $6.0$ | $7.0$ | $21$ | $30$ | $30$ | $34$ |
| $L = L_\mathsf{c}/2$ | NUM | $1.4$ | $1.6$ | $4.6$ | $6.3$ | $5.9$ | $11.2$ |
| | PT | $1.2$ | $1.4$ | $4.3$ | $6.0$ | $6.0$ | $6.7$ |




- Propagation constant gradients with respect to geometry alterations can be estimated by line integrals of the mode fields along relevant dielectric interfaces.
- Comparison of gradients from equivalent boundary movements, and comparison with conventionally computed derivatives provide a self consistency test for the underlying mode solver.
- Accuracy is sufficient to yield reasonable tolerance estimates for our simple coupler, from a single mode analysis only.
- With the geometry perturbation formulas and analogous expressions for refractive indices and wavelength, derivatives of the propagation constants with respect to all relevant parameters are at hand. Regarding the automated optimization of a device geometry, this should reduce the computational effort significantly.
- Guidelines for geometry tolerant devices:

$$\Delta q = \frac{2\sqrt{\Delta P}}{L|\partial_q \beta_\mathsf{s} - \partial_q \beta_\mathsf{a}|}\,, \quad \delta\beta = \frac{k_0^2}{2\beta} \frac{(\epsilon^- - \epsilon^+) \int_{y_0}^{y_1} |E_y|^2(x_\mathsf{b}, y)\,\mathsf{d}y}{\iint |E_y|^2 \mathsf{d}x\,\mathsf{d}y}\,\delta x\,.$$

  The total length should be kept short, intensities on dielectric surfaces and refractive index contrasts must be low, and only modes with similar profiles should interfere.